Dependence Concepts for Multivariate Spatial and Temporal Models: Part I and Part II

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Table of contents

Section 1:

* General Concepts of Dependence Part I

* Examples of dependence modelling in Spatial and Temporal State-Space Models

Table of contents

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MODEL BASED CHARACTERIZATIONS OF DEPENDENCE:

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- Following this work, the term copula was first coined as a mathematical concept in Abel Sklar's theorem [Sklar, 1959]
  $\Rightarrow$ showed that one-dimensional distributions can be joined by a copula function to form multivariate distributions.

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- C is quasi-monotone on its support $[0,1]^d$ i.e. for every hyperrectangle $B = \prod_{i=1}^{d} [x_i, y_i] \subseteq [0,1]^d$ the C-volume of B is non-negative.

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- In addition for every $\boldsymbol{a}$ and $\boldsymbol{b}$ in $[0,1]^d$, such that for each $a_i < b_i$ for all $i \in \{1, 2, \ldots, n\}$ the condition on the volume for copula C is satisfied: $V_C([\boldsymbol{a}, \boldsymbol{b}]) \geq 0$.

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  - NOTE: *The volume of an d-box is given by*

$$V_C([\boldsymbol{a}, \boldsymbol{b}]) = \sum sgn(\boldsymbol{v}) C(\boldsymbol{v})$$
$$= \triangle_{a_1}^{b_1} \triangle_{a_2}^{b_2} \cdots \triangle_{a_d}^{b_d} C(\boldsymbol{v})$$

where the sum is taken over all vertices $\boldsymbol{v}$ of the d-box $[\boldsymbol{a}, \boldsymbol{b}]$ and $sgn(\boldsymbol{v}) = 1$ if $v_k = a_k$ for an even number of k's of $sgn(\boldsymbol{v}) = -1$ if $v_k = a_k$ for an odd number of k's. In addition one defines the notation

$$\triangle_{a_k}^{b_k} C(\boldsymbol{u}) = C(u_1, u_2, \ldots, u_{k-1}, b_k, u_{k+1}, \ldots, u_d) - C(u_1, u_2, \ldots, u_{k-1}, a_k, u_{k+1}, \ldots, u_d).$$

Copula: consider random vector $\boldsymbol{X} \in \mathbb{R}^d$ with continuous distribution F. Then to every $\boldsymbol{X}$ one can associate a d-copula $C : [0,1]^d \mapsto [0,1]$, defined by

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Survival Copula: the survival copula is defined as follows

$$\begin{aligned}
\mathbb{P}\mathrm{r}[X_1 > x_1, X_2 > x_2] &= \overline{F}(x_1, x_2) \\
&= 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F(X_1, X_2) \\
&= \overline{F}_{X_1}(x_1) + \overline{F}_{X_2}(x_2) - 1 + C(F_{X_1}(x_1), F_{X_2}(x_2)) \\
&= \overline{F}_{X_1}(x_1) + \overline{F}_{X_2}(x_2) - 1 + C\left(1 - \overline{F}_{X_1}(x_1), 1 - \overline{F}_{X_2}(x_2)\right)
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\end{aligned}$$

Hence, one can define for instance in $d = 2$ the mapping $\widetilde{C} : [0,1]^2 \mapsto [0,1]$ by

$$\widetilde{C}(1-u, 1-u) = 1 - 2u - C(u,u)$$

to be the survival copula of C i.e. $\overline{F}(x_1, x_2) = \widetilde{C}(\overline{F}_{X_1}(x_1), \overline{F}_{X_2}(x_2))$

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- For all d-copula distributions $C \leq M^d$ and M^d can be thought of as a state of 'maximal concordance'.

Note: for $d \geq 3$ the function W^d is not strictly a copula, this can be seen by calculating $W^d\left([1/2, 1] \times [1/2, 1] \times \cdots \times [1/2, 1]\right)$ which may not produce $V_C\left([\boldsymbol{a}, \boldsymbol{b}]\right) \geq 0$.

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$$V_C\left([\boldsymbol{a},\boldsymbol{b}]\right) = \sum \operatorname{sgn}(\boldsymbol{v}) C(\boldsymbol{v}) = \triangle_{a_1}^{b_1} \triangle_{a_2}^{b_2} \cdots \triangle_{a_d}^{b_d} C(\boldsymbol{v})$$

where the sum is taken over all vertices $\boldsymbol{v}$ of the n-box $[\boldsymbol{a},\boldsymbol{b}]$ and $\operatorname{sgn}(\boldsymbol{v}) = 1$ if $v_k = a_k$ for an even number of k's of $\operatorname{sgn}(\boldsymbol{v}) = -1$ if $v_k = a_k$ for an odd number of k's and we used

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Applying this to the copula W^d for the d-box $[1/2,1]^d$ produces

$$\begin{aligned}
W^d\left([1/2,1]^d\right) &= \max\{1+1+\ldots+1-d+1,0\} \\
&- d\max\{1/2+1+\ldots+1-d+1,0\} \\
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$$C(\boldsymbol{u}) = W^d(\boldsymbol{u}). \tag{1}$$

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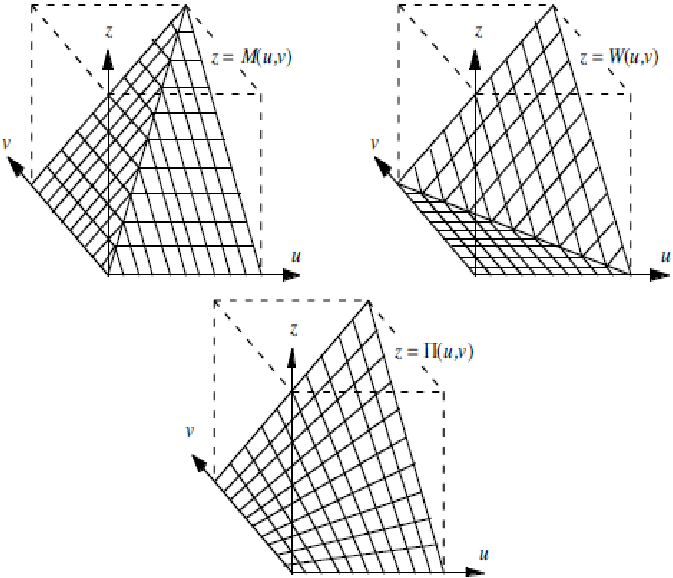
z
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Copulas and Transformations

Strictly Increasing Transformations

If $X_1, X_2, \dots, X_d$ are continuous r.v.'s with copula $C_{X_1,X_2,\dots,X_d}$. Then if $T_1(X_1), T_2(X_2), \dots, T_d(X_d)$ are strictly increasing on $\text{Ran}(X_1), \text{Ran}(X_2), \dots, \text{Ran}(X_d)$, then $C_{T_1(X_1),T_2(X_2),\dots,T_d(X_d)} = C_{X_1,X_2,\dots,X_d}$.

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- Consider marginal distributions $F_1, \ldots, F_d$ for continuous r.v.'s $X_1, \ldots, X_d$ and joint copula $C_{X_1,X_2,\ldots,X_d}$

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- $T_i(\cdot)$ is strictly increasing for each i, hence

$$G_i(x) = \mathbb{P}\text{r}\left(T_i(X_i) \leq x\right) = \mathbb{P}\text{r}\left(X_i \leq T_i^{-1}(x)\right) = F_i\left(T_i^{-1}(x)\right) \qquad (2)$$

for any $x \in \text{Ran}(X_i)$, hence one can show PTO

Copulas and Transformations

Proof Cont.:

$$
\begin{aligned}
& C_{T_1(X_1),T_2(X_2),\ldots,T_d(X_d)}\left(G_1(x_1),\ldots,G_d(x_d)\right) \\
& = \mathbb{P}\mathrm{r}\left(T_1(X_1)\leq x_1,\ldots,T_d(X_d)\leq x_d\right) \\
& = \mathbb{P}\mathrm{r}\left(X_1\leq T_1^{-1}(x_1),\ldots,X_d\leq T_d^{-1}(x_d)\right) \\
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& = C_{X_1,X_2,\ldots,X_d}\left(G_1(x_1),\ldots,G_d(x_d)\right)
\end{aligned} \tag{3}
$$

Since $X_1,\ldots,X_d$ are continous, $\mathrm{Ran}G_1 = \ldots \mathrm{Ran}G_d = [0,1]$. Hence it follows that $C_{T_1(X_1),T_2(X_2),\ldots,T_d(X_d)} = C_{X_1,X_2,\ldots,X_d}$ on $[0,1]^d$.

Copulas and Transformations

Strictly Monotone Transformations

If X_1 and X_2 are continuous r.v.'s with copula C_{X_1,X_2}. Then if $T_1(X_1)$ and $T_2(X_2)$ are strictly monotone on $\text{Ran}(X_1)$ and $\text{Ran}(X_2)$, then:

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MOST GENERAL APPROACH TO COPULA SIMULATION (SAMPLING)

- Consider general d-copula C, let the k-dim marginals of C be given by

$$C_k(u_1, \ldots, u_k) = C(u_1, \ldots, u_k, 1, \ldots, 1), \; k = 2, \ldots, d-1, \tag{4}$$

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- Let $U_1, \ldots, U_d$ have joint distribution C. Then the conditional distribution of U_k given $U_1, \ldots, U_{k-1}$ is given by

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Simulation

Step 1 Simulate a random variate u_1 from $U(0,1)$

Step 2 Simulate a random variate u_2 from $C_2(\cdot|u_1)$

$\vdots$

Step d Simulate a random variate u_d from $C_d(\cdot|u_1, \ldots, u_{d-1})$

Table of contents

Dependence Concepts Discussed:

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Beyond Linear Dependence

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Definition: Stochastic Ordering

Stochastic ordering (partial ordering) allows one to compare two random variables X_1 and X_2 and is characterized by $X_1 \preceq X_2$ (or $X_1 \leq_{st} X_2$) if and only if

$$\overline{F}_{X_1}(x) \leq \overline{F}_{X_2}(x), \quad \forall x.$$

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The following are all equivalent definitions:

- $X_1 \leq_{st} X_2 \Leftrightarrow F_{X_1}(x) \geq F_{X_2}(x), \quad \forall x.$
- $X_1 \leq_{st} X_2 \Leftrightarrow \mathbb{P}\mathrm{r}\left[X_1 \geq x\right] \leq \mathbb{P}\mathrm{r}\left[X_2 \geq x\right], \quad \forall x.$
- $X_1 \leq_{st} X_2 \Leftrightarrow \mathbb{E}_{X_1}\left[f(x)\right] \geq \mathbb{E}_{X_2}\left[f(x)\right]$, for all non-decreasing functions f.

Beyond Linear Dependence: Stochastic Ordering

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One can use the idea of partial stochastic orderings to define: Right Tail Decreasing, Left Tail Increasing, Left Tail Decreasing, Stochastically Decreasing and Regression Dependence as will be shown...

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Consider random variables $\{X_i\}_{i \geq 1}$. The sequence is lower or upper negatively dependent as follows:

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$$\mathbb{Pr}\left[X_1 \leq x_1, X_2 \leq x_2, \ldots, X_d \leq x_d\right] \leq \prod_{i=1}^{d} \mathbb{Pr}\left[X_i < x_i\right]$$

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- **Upper Negative Dependence:** A sequence of loss random variables are UND if for each $d \geq 1$ and all $X_1, X_2, \ldots, X_d$ one has

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- **Negative Dependence:** A sequence of loss random variables are ND if for each $d \geq 1$ and all $X_1, X_2, \ldots, X_d$ they satisfy that they are both LND and UND.

[Block et al. 1982], [Ghosh et al. 1981]

LND and UND versus Negative Association

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whenever f_1 and f_2 are increasing functions. [Joag et al 1983]

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Definition: Multivariate Positive Association

A d-vector $\{X_1, \ldots, X_d\}$ is PA if the inequality

$$\mathbb{E}\left[f_1\left(X_1, \ldots, X_d\right), f_2\left(X_1, \ldots, X_d\right)\right] \geq \mathbb{E}\left[f_1\left(X_1, \ldots, X_d\right)\right]\mathbb{E}\left[f_2\left(X_1, \ldots, X_d\right)\right]$$

holds for all real-valued f_1 and f_2 which are increasing. [Joe, 1997]

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Note: if X_i and X_j are PQD then one has $C\left(F_{X_i}(x), F_{X_j}(y)\right) \geq F_{X_i}(x) F_{X_j}(y)$ for all $\left(F_{X_i}(x), F_{X_j}(y)\right)$ in the unit square.

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- Many examples of copula model families exist that satisfy quadrant dependence.
  Example: many totally ordered one-parameter families of copulas have subfamilies of PQD copulas and NQD copulas.
    - Example: the Mardia family, the Farlie-Gumbel-Morgenstein FGM family, the Ali-Mikhail-Haq AMH family, or the Frank Archimedean family satisfy that they are PQD for copula parameter $\rho \geq 0$ and NQD for $\rho \leq 0$ with $\rho = 0$ giving $C = \Pi$.

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One can relate notions of Quadrant and Orthant Dependence to model based characterizations in a number of ways.

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- C_1 is more **Positive Orthant Dependent** than C_2, or C_1 is more concordant than C_2 if for all $\boldsymbol{u} \in [0,1]^d$, both $C_1(\boldsymbol{u}) \geq C_2(\boldsymbol{u})$ and $\overline{C}_1(\boldsymbol{u}) \geq \overline{C}_2(\boldsymbol{u})$ holds.

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Beyond Linear Dependence: Tail Monotonicity

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One can now also observe that a stronger condition than Quadrant dependence is to require that for each $x \in \mathbb{R}$, the conditional distribution function $\mathbb{P}\mathrm{r}\left[X \leq x | Y \leq y\right]$ is a non-increasing function of y.

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Remark

This stronger condition leads to the notion of Tail Decreasing and Tail Increasing, [Esary and Proschan, 1972].

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- Analogously, negative dependence properties, known as left tail increasing and right tail decreasing, are defined by exchanging the words nonincreasing and nondecreasing. [Kimeldorf and Sampson, 1987]

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see [Shaked, 1977].

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A random vector $(X_1, \ldots, X_d)$ as comonotonic if its multivariate distribution satisfies

$$\mathbb{P}\mathrm{r}\left[X_1 \leq x_1, \ldots, X_d \leq x_d\right] = \min_{i \in \{1,\ldots,d\}} \mathbb{P}\mathrm{r}\left[X_i \leq x_i\right].$$

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- If a random vectors density is MTP2 then so are all of its marginal densities of order 2 and higher.
- IF the above inequality expression has its inequality sign reversed, then the density f is said to be multivariate reverse rule of order 2 (MRR2) which is a weak negative dependence concept. Unlike MTP2, the property of MRR2 is not closed under marginalization!

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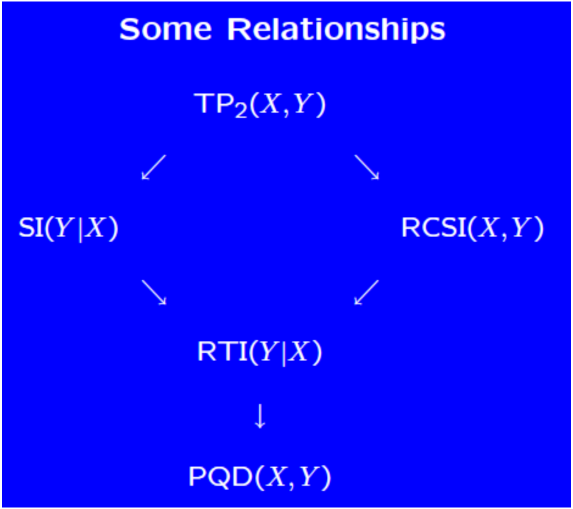
Some Relationships
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Section 2:

* General Concepts of Dependence Part II

* Measures of Dependence and Concordance

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Many measures of concordance are available!

Concordance and Dependence Measures

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$$r_n \kappa_d\left(C\right) = \kappa_{n+1}\left(E\right) + \kappa_{n+1}\left(E(1-u_1, u_2, \ldots, u_d)\right)$$

whenever E is an $(d+1)$-copula s.t. $C(u_1, \ldots, u_d) = E(1, u_1, \ldots, u_d)$.

Theorem: Properties of Concordance Measures Satisfying [Taylor, 2006] Axioms

Consider the d-copula that is permutation symmetric ie. $C^{\zeta} = C$ for all permuations ζ of $[0,1]^d$. Then for all measures of concordance κ and for all symmetries Ψ and ζ of $[0,1]^d$ one has

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Recall: symmetry length $|\cdot|$ corresponds to the number elementary reflections required to obtain it.

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Corollary

For all $d \geq 2$ and for all symmetries Ψ and ζ of $[0,1]^d$ such that $|\Psi| = |\zeta|$ or $|\Psi| + |\zeta| = d$ one has

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where M is the d-Frechet-Hoffding Upper Bound copula under permutation.

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The population Kendall's tau is the probability of concordance minus the probability of discordance, given for two random vectors (X_1, Y_1) and (X_2, Y_2) by

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Example: consider the well known measure of concordance versus discordance which is an association measure - Kendall's tau.

Kendall's Tau

The population Kendall's tau is the probability of concordance minus the probability of discordance, given for two random vectors (X_1, Y_1) and (X_2, Y_2) by

$$\tau = \mathbb{P}\text{r}\left[(X_1 - X_2)(Y_1 - Y_2) > 0\right] - \mathbb{P}\text{r}\left[(X_1 - X_2)(Y_1 - Y_2) < 0\right]$$

Relating Copulas to Concordance Measures of Association

Consider the concordance function κ quantifying the difference in probabilities of concordance and discordance for bi-variate random vectors (X_1, Y_1) and (X_2, Y_2).

- Assume X_1 and X_2 have common continuous marginal F_X
- Assume Y_1 and Y_2 have common continuous marginal F_Y
- Assume (X_1, Y_1) and (X_2, Y_2) have different copula C_1 and C_2 respectively.

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Concordance and Dependence Measures

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Recall: M^d - Frechet-Hoffding Upper-Bound; W^d - Frechet-Hoffding Lower-Bound; and Π^d - independence copula.

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- Due to this scale-invariance, rank correlations thus provide an approach for fitting copulae to data.

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Definition: Bivariate Spearman's Rank Correlation Coefficient

Consider two sets of order statistics $\left\{X_{(i,n)}\right\}_{i=1}^{d}$ and $\left\{Y_{(i,n)}\right\}_{i=1}^{d}$, then spearman's rank correlation is

$$\rho := \frac{\sum_{i=1}^{d}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_i (x_i - \bar{x})^2 (y_i - \bar{y})^2}}$$

where x_i, y_i are the ranks.

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Consider the n-copula given by C and the permuted copula C^{σ} according to

$$\rho_d(C) = \alpha_d \left(\int_{[0,1]^d} \left(C + C^{\sigma}\right) d\Pi^d - \frac{1}{2^{d-1}} \right)$$

where one has $\alpha_d = \frac{(d+1)2^{d-1}}{2^d - (d+1)}$ and Π^d is the d-Independence Copula.

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where med (X_i) is the median of random variable X_1, [Blomqvist, 1950].

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where $\mathrm{med}\left(X_i\right)$ is the median of random variable X_1, [Blomqvist, 1950].

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- The empirical version $\widehat{\rho}_\beta$ of Blomqvists beta is a suitably scaled version of the proportion of points whose components are either both smaller, or both larger, than their respective sample medians
- The computation of $\widehat{\rho}_\beta$ involves only $O(n)$ operations, as opposed to $O(n^2)$ for the empirical versions of Kendalls tau and Spearmans rho.

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Definition: Generalized Blomqvist's Beta

Consider an d-copula C, then the generalized Blomqvist's Beta is given by

$$\beta_d(C) = \alpha_d\left(C(\frac{1}{2}, \ldots, \frac{1}{2}) - \frac{1}{2^d}\right)$$

where $\alpha_d = \frac{2^d}{2^{d-1}-1}$

Intermediate Directional Dependence

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3-Copula ρ-Directional Dependence

Consider a random vector $\boldsymbol{X} = (X_1, X_2, X_3)$ with $\boldsymbol{X} \in \mathbb{R}^3$ and associated 3-dimensional copula $C_{\boldsymbol{X}}$. Then for any direction $(\alpha_1, \alpha_2, \alpha_3)$ characterised by the vector components $\alpha_i \in \{-1, 1\}$ for $i \in \{1, 2, 3\}$, one has the ρ-directional dependence given by

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$$\rho^{(\alpha_1,\alpha_2,\alpha_3)}_{X_1,X_2,X_3} = \frac{\alpha_1\alpha_2\rho_{X_1,X_x} + \alpha_2\alpha_3\rho_{X_2,X_3} + \alpha_3\alpha_1\rho_{X_3,X_1}}{3} + \alpha_1\alpha_2\alpha_3\frac{\rho^{+}_{X_1,X_2,X_3} - \rho^{-}_{X_1,X_2,X_3}}{2}$$

with pairwise Spearman's rho and

$$\rho^{+}_{X_1,X_2,X_3}(C_{\boldsymbol{X}}) = 8\int_{[0,1]^3} \overline{C}_{\boldsymbol{X}}(u,v,w)dudvdw - 1,$$

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Remark

The eight vectors which characterize directions $(\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_i \in \{-1, 1\}$ for $i \in \{1, 2, 3\}$ in $[0, 1]^3$ allow one to utilise the ρ-directional dependence to measure directional dependence in different quadrants.

Remark

The eight vectors which characterize directions $(\alpha_1, \alpha_2, \alpha_3)$ where $\alpha_i \in \{-1, 1\}$ for $i \in \{1, 2, 3\}$ in $[0, 1]^3$ allow one to utilise the ρ-directional dependence to measure directional dependence in different quadrants.

- Example: if $\rho_{\boldsymbol{X}}^{(-1,-1,1)}$ or $\rho_{\boldsymbol{X}}^{(1,1,-1)}$ are positive, then there will be positive dependence in the direction of $(-1, -1, 1)$ or $(1, 1, -1)$, hence one would expect large (small) values of X_1 and X_2 to occur with small (large) values of X_3, ie. $\rho_{X_1,X_2} > 0$ with $\rho_{X_1,X_3} < 0$ and $\rho_{X_2,X_3} < 0$.

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Extending the notions of concordance measure beyond linear relationships through model based characteristics has been done from first principles by [Taylor, 2007] in the multivariate setting extending [Scarsini, 1984]

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Consider X_1 and X_2 jointly distributed as symmetric α-Stable $S\alpha S$ with $\alpha \in (1, 2)$. Then the co-variation and co-difference are defined by

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where $z^{<p>} = |z|^p \mathrm{sgn}(z)$ and $\mathbb{S}^2$ is the unit 2-sphere defined by

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Note: Discussion on Copula and Spectral Measure Relationships Later!

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- If $\alpha = 2$ then co-difference, co-variation and covariance are related as follows:

$$\mathbb{C}\mathrm{ov}\left(X_1, X_2\right) = 2\mathrm{CV}\left(X_1, X_2\right) = \mathrm{CD}\left(X_1, X_2\right)$$

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Properties of Covariation and Codifference

- In contrast to the co-difference, the covariation is not symmetric in its arguments.
- If $\alpha > 1$ then the covariation induces a norm on the linear sub-space of jointly SαS random variables

$$||\boldsymbol{X}||_{\alpha} = \left[\mathrm{CV}\left(X_1, X_2\right)\right]^{1/\alpha}$$

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   However, it is interesting to question whether dependence properties still hold if focusing only on extremes of the distribution in any particular quadrant?

   For instance if the correlation between X_1 and X_2 is positive, is it reasonable to assume that the correlation between extreme values of X_1 and extreme values of X_2 will still be positive or even present at all ?

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Remark

Similar to rank correlations, the tail dependence coefficient is a simple scalar measure of dependence that depends on the copula not the marginals.

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- Analogously, the lower tail dependence coefficient of copula C is the upper tail dependence coefficient of the survival copula.
- λ_u and λ_l belong to the range $[0, 1]$, provided the limits exist.

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one can show the following features of upper tail dependence:

- The upper tail dependence satisfies the bound

$$c\lambda_u \leq \widehat{\lambda} \leq \min(c, \lambda_u)$$

with

$$\widehat{\lambda} = \lim_{x \to \infty} \frac{1 - F_{X_1}(x) - F_{X_2}(x) + C\left(F_{X_1}(x), F_{X_2}(x)\right)}{1 - F_{X_1}(x)}.$$

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- the following relationship between the maximum of a sum of two random variables and the tail dependence holds

$$\mathbb{P}\mathrm{r}\left[\max\left\{X_1, X_2\right\} > x\right] \sim \left(1 + c - \widehat{\lambda}\right) \overline{F}_{X_1}(x)$$

  and the tail result given by

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- The following worst case bounds can be obtained

$$\overline{F}_{X_1}(x) << \mathbb{P}\mathrm{r}\left[X_1 + X_2 > x\right] << (1 + c)\overline{F}_{X_1}\left(\frac{x}{2}\right).$$

Finally, one can also obtain the following upper and lower bounds for common marginals.

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Properties of Tail Dependence Coefficient Cont. II

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Properties of Tail Dependence Coefficient Cont. II

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$$\lambda_u \leq \liminf_{x \to \infty} \frac{\mathbb{P}\mathrm{r}\left[c_1 X_1 + c_2 X_2 > x\right]}{\mathbb{P}\mathrm{r}\left[X_1 > \frac{x}{c_1 + c_2}\right]},$$

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for constants c_1 and c_2 satisfying $y = \frac{c_1 x}{(c_1 + c_2)}$.

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Let $X = (X_1, ..., X_d)^T$ be a d-dimensional random vector with marginal distributions $F_1, ..., F_d$ and copula C.

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- [Coles, 1999] argues that using $\overline{\chi}$ in addition to a tail dependence measure gives a more complete summary of extremal dependence.

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$$\lim_{r\to\infty} \frac{\mathbb{P}\text{r}\left(\boldsymbol{X} \in \text{Cone}(A), ||\boldsymbol{X}|| > r\right)}{\mathbb{P}\text{r}\left(||\boldsymbol{X}|| > r\right)} = \lim_{r\to\infty} \mathbb{P}\text{r}\left(\boldsymbol{X} \in \text{Cone}(A) |\, ||\boldsymbol{X}|| > r\right) = \frac{\Gamma(A)}{\Gamma(\mathbb{S}_d)}$$

Example: Linking Orthant Extreme Dependence to Spectral Measures

Consider the bivariate example for the upper tail dependence:

$$\lambda_u = \lim_{u \uparrow 1} \mathbb{P}r\left(X_1 > F_{X_1}^{-1}(u) | X_2 > F_{X_2}^{-1}(u)\right) = \lim_{u \uparrow 1} \frac{1 - 2u + C(u,u)}{1-u}.$$

Recall that for a set $A \subset \mathbb{S}_d$ one can define the cone generated by A to be

$$\text{Cone}(A) = \left\{ \boldsymbol{x} \in \mathbb{R}^d : ||\boldsymbol{x}|| > 0, \frac{\boldsymbol{x}}{||\boldsymbol{x}||} \in A \right\} = \{ r\boldsymbol{a} : r > 0, \boldsymbol{a} \in A \},$$

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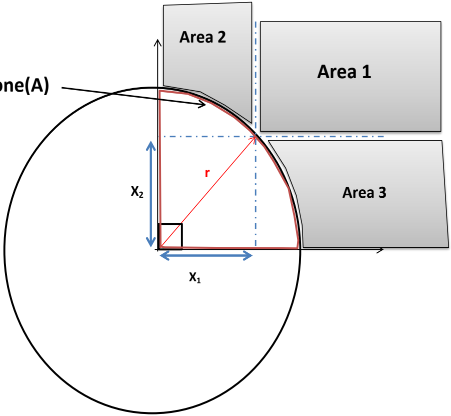
How do we link tail dependence (e.g. λ_u) to the Spectral Measure $\Gamma(\cdot)$?

First: Observe that if one selects the set A to be the upper right quadrant mapped out by the angle $[0, \pi/2]$ that makes the cone Cone(A) correspond to an arc on the top right quadrant, then one has the following relationship:

$U(r) = F(X_1)$

$U(r) = F(X_2)$



Rewrite these probabilities for Area 1, Area 2 and Area 3.

$$\mathbb{P}\mathrm{r}\left(\boldsymbol{X} \in \mathrm{Cone}(A) \,|\, ||\boldsymbol{X}|| > r\right) = \underbrace{\mathbb{P}r\left(X_1 > x_1, X_2 > x_2\right)}_{\text{Area 1}}$$
$$+ \underbrace{\left[\mathbb{P}r\left(X_1 < x_1, X_2 > x_2\right) - \mathbb{P}r\left(X_1 < x_1, X_2 \in [x_2, r] \,|\, ||\boldsymbol{X}|| < r\right)\right]}_{\text{Area 2}}$$
$$+ \underbrace{\left[\mathbb{P}r\left(X_1 > x_1, X_2 < x_2\right) - \mathbb{P}r\left(X_1 \in [x_1, r], X_2 < x_2 \,|\, ||\boldsymbol{X}|| < r\right)\right]}_{\text{Area 3}}$$

- If we now take the limit on both sides, we will be able to obtain the link between the tail dependence of the random vector $\boldsymbol{X}$ and the spectral measure $\Gamma(\cdot)$.
- Next we see some examples and special cases of results

Beyond Linear Dependence:

Linking Regional Dependence to Model Properties!

- **Example:** consider the class of random vectors $\boldsymbol{X} \in \mathbb{R}^d$ which have an infinitely divisible law.

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Definition: Levy-Khintchine Formula

A probabilty law μ of a real-valued random vector is inifinitely divisible with characteristic exponent Ψ, given by

$$\int_{\mathbb{R}^d} \exp\left(i < \boldsymbol{\theta}, \boldsymbol{x} >\right) \mu(d\boldsymbol{x}) = \exp\left(-\Psi(\boldsymbol{\theta})\right), \text{ for } \boldsymbol{\theta} \in \mathbb{R}^d$$

iff there exists a triple $(a, \Sigma, W(d\boldsymbol{x}))$, where $\boldsymbol{a} \in \mathbb{R}^d$, $\Sigma \in SPD(\mathbb{R}^d)$ and $W(d\boldsymbol{x})$ is a measure concentrated on $\mathbb{R}^d \setminus \{0\}$ satisfying $\int_{\mathbb{R}^d} \left(1 \wedge ||\boldsymbol{x}||^2\right) W(\boldsymbol{dx}) < \infty$, s.t.

$$\Psi(\boldsymbol{\theta}) = i < \boldsymbol{a}, \boldsymbol{\theta} > + \frac{1}{2}\boldsymbol{\theta}\Sigma\boldsymbol{\theta}^T + \int_{\mathbb{R}^d} \left(1 - e^{i<\boldsymbol{\theta},\boldsymbol{x}>} + i < \boldsymbol{\theta}, \boldsymbol{x} > \mathbb{I}_{||\boldsymbol{x}||<1}\right) W(d\boldsymbol{x})$$

UCL

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- Measure $W(d\boldsymbol{x})$ is known as the Levy measure and it is unique.
- Spectral measure can be shown to be directly linked to aspects of dependence of the random vector.

Linking Regional Dependence to Model Properties!

One can map between the spectral measure $W(d\boldsymbol{x})$ defined on $\mathbb{R}^d$ and the spectral measure in polar co-ordinates on unit hyper-sphere $\Gamma(d\boldsymbol{s})$ on $\mathbb{S}_d$ as shown in the pure-jump process setting of Tempered Stable models, see e.g. [Rosinski, 2007].

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The mass that $\Gamma(\cdot)$ assigns to A determines the tail behavior of $\boldsymbol{X}$ in the direction of A.

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$$\lim_{t\to\infty} t\mathbb{P}\mathrm{r}\left(||\boldsymbol{X}|| > rb(t), \frac{\boldsymbol{X}}{||\boldsymbol{X}||} \in B\right) = qr^{-\beta}\mu(B) \tag{5}$$

for any Borel set $B \subset \left\{(x_1, \ldots, x_d) \in \mathbb{R}^d \middle| \; ||\boldsymbol{x}|| = 1\right\}$. Then $\boldsymbol{X}$ is said to be $MRV_d(-\beta)$.

Remark

It can then be shown [Barbe, 2006] and [Resnick, 2004] that for $\boldsymbol{X} \in MRV_d(-\beta)$ for $\beta > 0$ one has

$$q\left(\beta, ||\cdot||\right) = \lim_{x \to \infty} \frac{\mathbb{P}\mathrm{r}\left(||\boldsymbol{X}|| > x\right)}{\mathbb{P}\mathrm{r}(X_1 > x)} > 0 \tag{6}$$

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Lemma: MVR Expressed Via Quantiles

If $\boldsymbol{X} = (X_1, \ldots, X_d) \in MRV_d(-\beta)$ with $\beta > 0$ and identically distributed marginals. Then for a measurable function $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$,

$$\lim_{x \to \infty} \frac{\mathbb{P}\mathrm{r}\left(\varphi(\boldsymbol{X}) > x\right)}{\mathbb{P}\mathrm{r}\left(X_1 > x\right)} = q_\varphi \in (0, \infty) \tag{7}$$

which implies that for quantile functions Q at level α one has

$$\lim_{\alpha \uparrow 1} \frac{Q_\alpha\left(\varphi(\boldsymbol{X})\right)}{Q_\alpha(X_1)} = q_\varphi \tag{8}$$

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Then one can show the following relationship between such a measure and the limiting behaviour of a MRV random vector:

$$\lim_{t \to \infty} t\mathbb{P}\text{r}\left(\frac{\boldsymbol{X}}{b(t)} \in B \right) = \mu_\beta(B) \tag{10}$$

To further relate

$$\lim_{t\to\infty} t\mathbb{P}\mathrm{r}\left(\frac{\boldsymbol{X}}{b(t)} \in B\right) = \mu_\beta(B) \tag{11}$$

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By the definition of MVR one has the constant q (depending on β and norm $||\cdot||$) given by:

$$q(\beta, ||\cdot||)r^{-\beta}\mu(G) = \nu_\beta\left\{\boldsymbol{x} \in [0,\infty]^d \middle|\ ||\boldsymbol{x}|| > r, \frac{\boldsymbol{x}}{||\boldsymbol{x}||} \in G\right\} \tag{12}$$

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Theorem: MVR and Spectral Measure Representation

Let the $\mathbb{R}_+^d$ valued random vector $\boldsymbol{X}$ with i.i.d. marginals satisfy $\boldsymbol{X} \in MVR(-\beta)$ with $\beta > 0$, then

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Table of contents

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Fisheries Economics Example: SSM and Dependence

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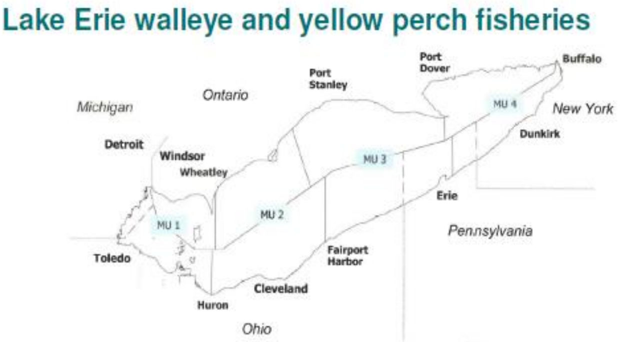
[Hossack, Peters and Ludsin, 2014] demonstrate that such economic decisions as stock quota must be set with interspecific species, spatial dependence and environmental factors taken into consideration! (dependence)



Walleye

Yellow Perch

Marginal process model for a given stock[1]

Based on Schaefer surplus production model:

$$\log X_{t+1}^{(s)} = \log\left[X_t^{(s)} + r^{(s)} X_t^{(s)}\left(1 - \frac{X_t^{(s)}}{k^{(s)}}\right) - H_t^{(s)}\right] + \epsilon_{t+1}^{(s)},$$

where $\epsilon_t^{(s)} \overset{i.i.d.}{\sim} \mathcal{N}\left(0, \left(\sigma_\epsilon^{(s)}\right)^2\right)$, and,

$X_t^{(s)}$: latent stock size of stock s in year t

$H_t^{(s)}$: total harvest of stock s in year t

$r^{(s)}$, $k^{(s)}$: growth rate parameters

[1] Hilborn, R. and Walters, C. J. (1992). *Quantitative Fisheries Stock Assessment*. Chapman and Hall

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Fisheries Economics Example: SSM and Dependence

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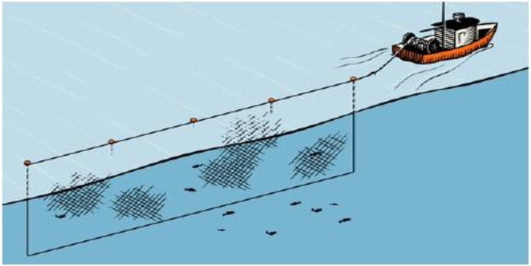
In population biology, carrying capacity is defined as the environment's maximal load, which is different from the concept of population equilibrium.

Key fisheries challenge: 3 sources of uncertainty

observation error: catch per unit effort (CPUE) data
catchability: how fisheries interact with fish stocks
process model: stock dynamics



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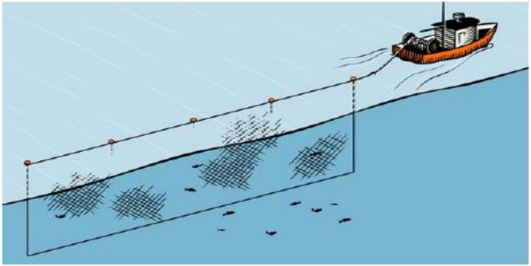

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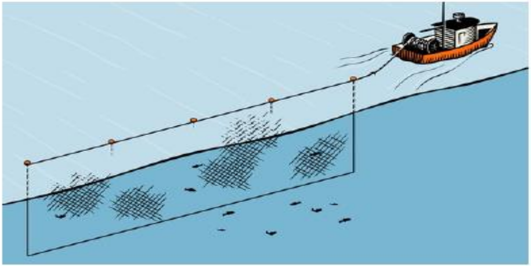

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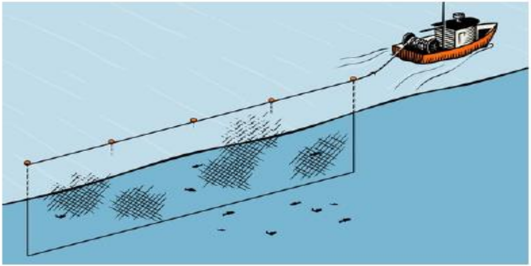

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and $A_t^{(s,f,m)}$ represents the time, space and species varying catchabilities.

The relationship between fish abundance and efficiency of fishing gear is catchability $\Rightarrow$ Catachability measures interaction between the resource and the predation effort.

In more detail - the catchabilities have the following structure:

$$A_t^{(s,f,m)} \sim N\left(\left[a^{(s,f)} + \beta^{(s,f)}\right]\mathbb{I}, \nu^{(s,f)}\mathbb{R}(\rho_a^{(s,f)})\right) \tag{17}$$

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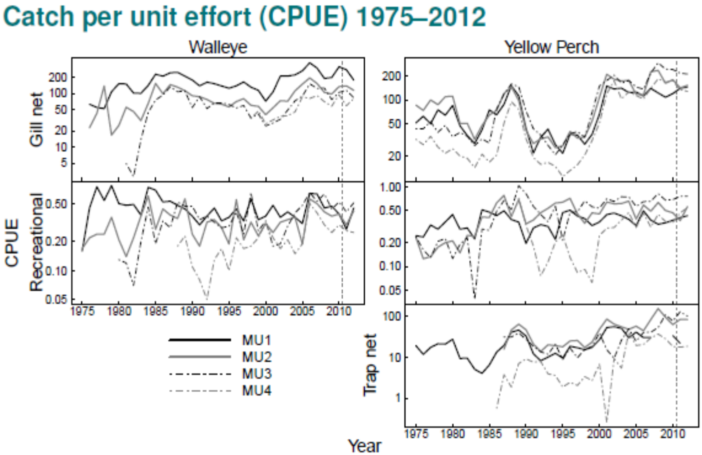
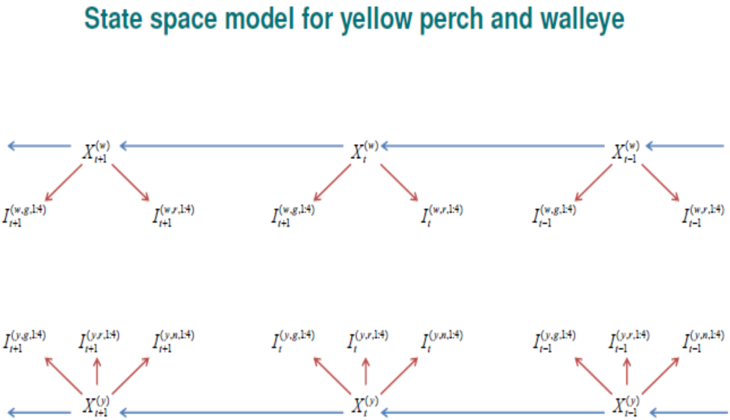


- interannual variability due to changes in both environment and fisheries management

The Independent Latent Process SSM - for Stock sizes given CPUE's



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Such features may jointly affect the recruitment or natural mortality of all the relevant stocks or species - which can be better understood through incorporation of dependence structures in the SSM.

The SSM - for Stock sizes given CPUE's

Interspecific process uncertainty via copula $c(\cdot)$

Copula joins marginal process models:

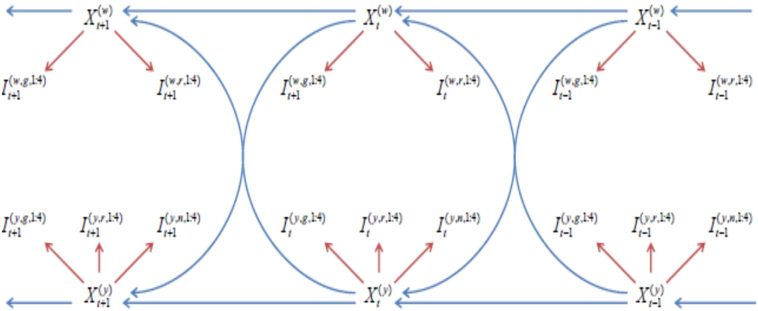
$$\left.\begin{aligned}\log X_{t+1}^{(w)} &= \log\left[X_t^{(w)} + r^{(w)} X_t^{(w)}\left(1 - \frac{X_t^{(w)}}{k^{(w)}}\right) - H_t^{(w)}\right] + \epsilon_{t+1}^{(w)} \\ \log X_{t+1}^{(y)} &= \log\left[X_t^{(y)} + r^{(y)} X_t^{(y)}\left(1 - \frac{X_t^{(y)}}{k^{(y)}}\right) - H_t^{(y)}\right] + \epsilon_{t+1}^{(y)}\end{aligned}\right\} c(\cdot)$$

Ecological interpretation for $c(\cdot)$: annual recruitment or natural mortality

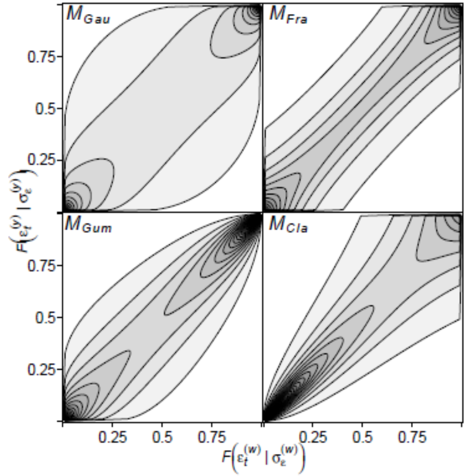
- Gaussian copula M_{Gau} – linearly correlated
- Frank copula M_{Fra} – strongly associated in typical years
- Gumbel copula M_{Gum} – coincident and rare recruitment spikes
- Clayton copula M_{Cla} – coincident and rare mortality spikes

The Copula Dependent SSM

State space model: interspecific dependence



Some Results of Estimations

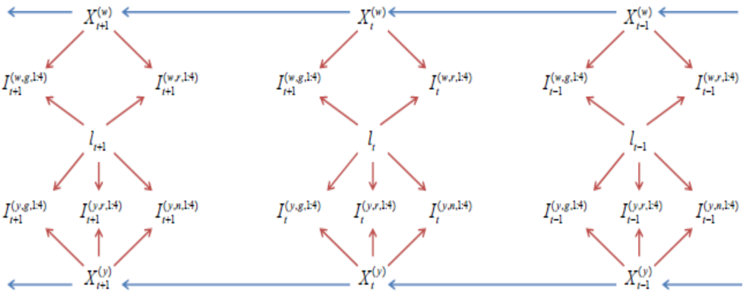


- Copula predictive density: $c(\cdot|\boldsymbol{I}_{1:T}, M_k) = \int c(\cdot|\rho_\epsilon, M_k)p(\boldsymbol{\theta}|\boldsymbol{I}_{1:T}, M_k)d\boldsymbol{\theta}$

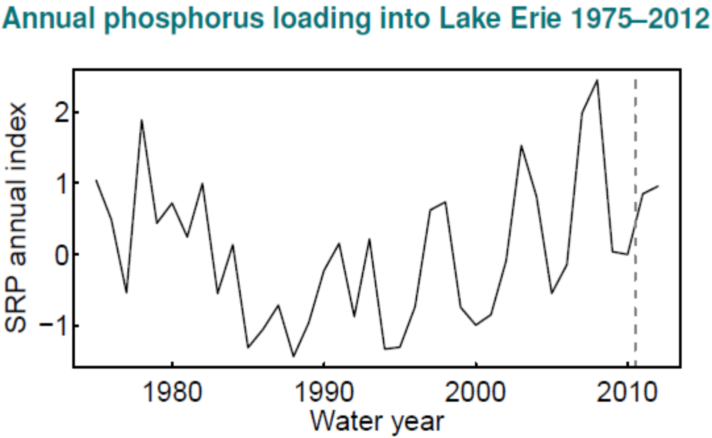
Alternative Dependence Stuctures in SSM for Stock sizes given CPUE's

Temporal dependence with dependent catchabilities

Dependence induced by common factor l_t:



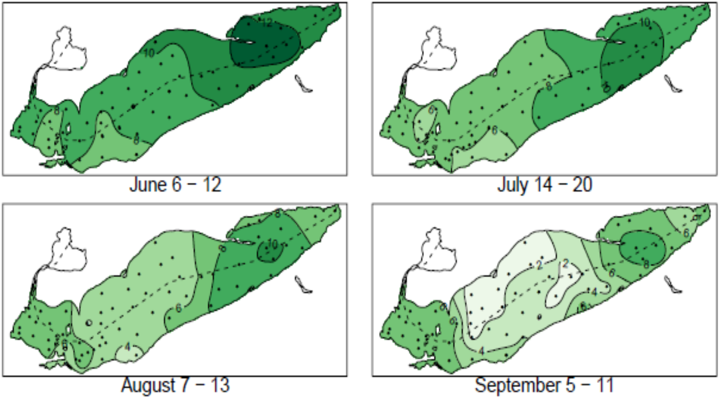
Example of relevant common factor:



- linked to hypoxia formation in Lake Erie (Rucinski et al. 2010, Daloğlu et al. 2012)

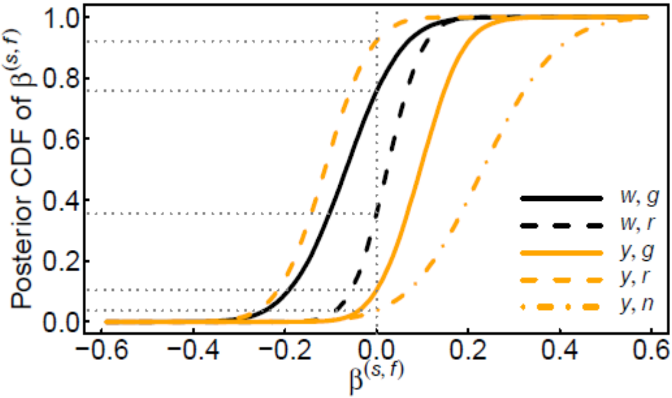
Fisheries Economics Example: SSM and Dependence

Hypoxia in Lake Erie, June–September 2005

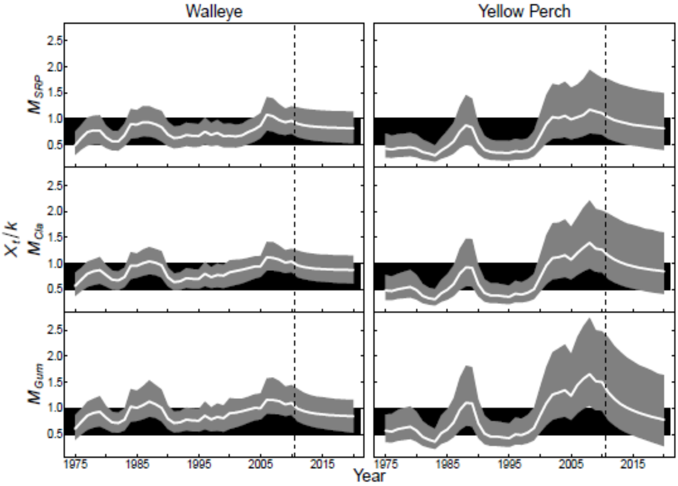


- hypoxia could spatially compress fish and increase catchability

Catchability and soluble reactive phosphorus (M_{SRP})



- Yellow perch trap net fishery positive with 0.96 probability
- Yellow perch recreational fishery negative with 0.90 probability



- tail dependence affects latent path space