Event Detection in Wireless Sensor Networks in Random Spatial Sensors Deployments

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Outline

1. Spatial Point Processes
2. Wireless sensor network system model
3. Estimation goals
4. Algorithms development
5. Simulations
6. Conclusions

Spatial Point Processes

Definition (Poisson Point Process)

A PPP is defined using the following two properties:

- The number of points in disjoint subsets is independent.
- For any bounded subset, the number of points follows a Poisson distribution.

$$\Pr\left(N\left(B_i\right) = k_i, 1 \leq i < n\right) = \prod_i \exp^{\left(\int_{B_i} \lambda(x) \mathrm{d}x\right)} \frac{\left(\int_{B_i} \lambda\left(x\right) \mathrm{d}x\right)^{k_i}}{k_i!}$$

For homogeneous PPP we have that $\int_{B_i} \lambda\left(x\right) \mathrm{d}x = \lambda \left\| B_i \right\|$.

Definition (Finite Binomial Point Process (FBPP))

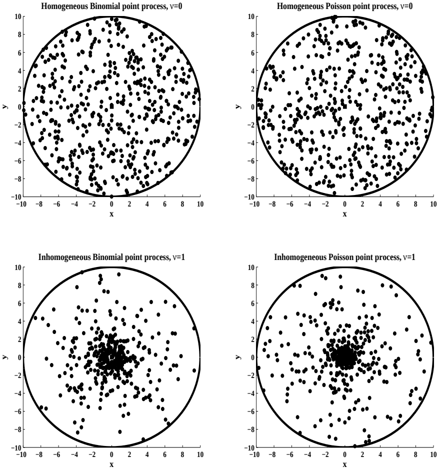
A Finite Binomial Point Process is defined by considering a fixed number of n points at random locations in a bounded region $W \subset \mathbb{R}^2$. Define by $X_1, \ldots, X_n$ the i.i.d. the random locations with the intensity of the number of points in a small region around any location $\mathbf{x}$ defined to be $\lambda(\mathbf{x})$. This produces a probability density of each X_i given by

$$f_X(x) = \begin{cases} \frac{1}{\lambda(W)}, & \text{if } x \in W, \\ 0, & \text{otherwise}, \end{cases}$$

where $\lambda(W)$ denotes the area of W. Each random point X_i is uniformly distributed in W so that for a bounded set $B \in \mathbb{R}^2$ on has the distribution

$$\mathbb{P}\mathrm{r}\left(X_i \in B\right) = \int_B f_X(x)dx = \frac{\lambda(B \cap W)}{\lambda(W)}.$$

Example



Homogeneous Binomial point process, ν=0
Homogeneous Poisson point process, ν=0
Inhomogeneous Binomial point process, ν=1
Inhomogeneous Poisson point process, ν=1
x
y

System Model:
sensor network for event detection

System Model

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- A3 Sensors are deployed and their locations follow either a BPP or a PPP deployment in a 2 dimensional circular region with radius R. The spatial density of the sensors is given by $\lambda(\mathbf{x}) = \mathbf{x}^{-\nu}, \nu \neq 0$.
- A4 The random unknown location of the k-th sensor ($k = \{1, \cdots, N\}$) is $\mathbf{X}_k = [X_k, Y_k]$.

A5 The amount of energy the k-th sensors measures is inversely proportional to the Euclidean distance between the target and the sensor and is given by $\sqrt{p_0} R_k^{-\alpha/2}$. The random variable R_k represents the random distance between the k-th sensor and the target.

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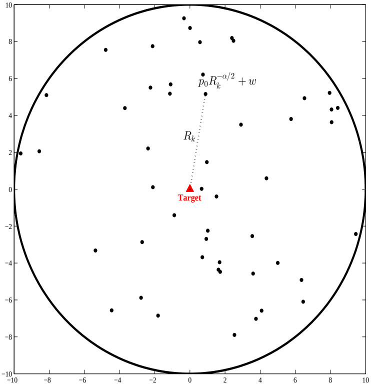
A6 Each senors transmits its observation over perfect channels to the gateway (GW) via a shared medium. The observed signal at the GW in the l-th time slot ($l = \{1, \cdots, L\}$) is a linear combination of all the signals given by:

$$\begin{cases} \mathcal{H}_0 : Y_l = W_l \\ \mathcal{H}_1 : Y_l = \sum_{k=1}^{N} \sqrt{p_0} {R_k}^{-\alpha/2} + W_l, \end{cases}$$

where W_l is the i.i.d additive Gaussian noise $\mathcal{N}(0, \sigma^2_{W_l})$. The parameter α is the path-loss coefficient.

Is or isn't there a target at $\mathbf{x}_0$?



Null hypothesis: go hiking

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Alternative hypothesis:
Run away!

Problem statement: the optimal detector

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The optimal decision rule is a threshold test based on the likelihood ratio given by:

$$\Lambda\left(Y_{1:L}\right) \triangleq \frac{p(Y_{1:L}|\mathbf{x}_s, \mathcal{H}_0)}{p\left(Y_{1:L}|\mathbf{x}_s, \mathcal{H}_1\right)} \underset{\mathcal{H}_1}{\overset{\mathcal{H}_0}{\gtrless}} \gamma,$$

where the threshold γ can be set to assure a fixed system false-alarm rate under the Neyman-Pearson approach or can be chosen to minimize the overall probability of error under the Bayesian approach. We can decompose the full marginals under each hypothesis, $p\left(Y_{1:L}|\mathbf{x}_s, \mathcal{H}_k\right)$, $k = 0, 1$, as

$$p(Y_{1:L}|\mathbf{x}_s, \mathcal{H}_k) = \prod_{l=1}^{L} p(Y_l|\mathbf{x}_s, \mathcal{H}_k).$$

Deriving the marginal likelihoods

Solution

The easy part: marginal likelihood under the Null

$$\mathcal{H}_0 : Y_l = W_l$$

The marginal likeloihood:

$$\mathcal{H}_0 : p(Y_{1:L}|\mathbf{x}_s, \mathcal{H}_0) = \prod_{l=1}^{L} \mathcal{N}(Y_l; 0, \sigma_W^2)$$

Solution

The difficult part: marginal likelihood under the Alternative

$$\mathcal{H}_1 : Y_l = \sum_{k=1}^{N} \sqrt{p_0} {R_k}^{-\alpha/2} + W_l.$$

The marginal likelihood:

$$\mathcal{H}_1 : p(Y_{1:L} | \mathbf{x}_s, \mathcal{H}_1) = ?$$

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3. Solve the N-fold convolution.

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3. Solve the N-fold convolution.
4. Solve the convolution with W_l.

Step 1: target to k-th sensor random distance density

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Theorem

The density of the Euclidean distance between the k-th sensor and the target, R_k is:

1. *BPP deployment:*

$$f_{R_k}\left(r|\mathbf{x}_s,\mathcal{H}_1\right)=\frac{(2-\nu)\,\Gamma\left(k+\frac{1-\nu}{2-\nu}\right)\Gamma\left(N_B+1\right)}{R\,\Gamma\left(k\right)\Gamma\left(N_B+\frac{1-\nu}{2-\nu}+1\right)}\beta\left(\left(\frac{r}{R}\right)^{2-\nu};k+\frac{1-\nu}{2-\nu},N_B-k+1\right).$$

2. *PPP deployment:*

$$f_{R_k}\left(r|\mathbf{x}_s,n=N_P,\mathcal{H}_1\right)=\frac{(2\pi)^k}{\Gamma\left(k\right)(2-\nu)^{k-1}}r^{(2-\nu)k-1}\exp\left(-\frac{2\pi r^{2-\nu}}{2-\nu}\right),$$

where $\beta\left(x,\alpha,\beta\right):=\frac{1}{B(\alpha,\beta)}x^{\alpha-1}\left(1-x\right)^{\beta-1}$ *is the* β *distribution and* $\Gamma\left(n\right):=\left(n-1\right)!$ *is the Gamma function.*

Step 2: transformed distance density

$$p\left(R_k^{-\alpha/2}\right)$$

Step 2: transformed distance density $p\left(R_k^{-\alpha/2}\right)$

Lemma

The density $f_{Z_k}\left(z|\mathbf{x}_s,\mathcal{H}_1\right) = f_{Z_k}\left(r^{-\alpha/2}|\mathbf{x}_s,\mathcal{H}_1\right)$ *is given by:*

1. *BPP deployment:*

$$f_{Z_k}\left(z|\mathbf{x}_s,\mathcal{H}_1\right) = \frac{2\left(2-\nu\right)}{\alpha R}\frac{\Gamma\left(k+\frac{1-\nu}{2-\nu}\right)\Gamma\left(N_B+1\right)}{\Gamma\left(k\right)\Gamma\left(N_B+\frac{1-\nu}{2-\nu}+1\right)} \times \beta\left(\left(\frac{z^{-2/\alpha}}{R}\right)^{2-\nu};k+\frac{1-\nu}{2-\nu},N_B-k+1\right)z^{-2/\alpha-1}$$

2. *PPP deployment:*

$$f_{Z_k}\left(z|\mathbf{x}_s,n=N_P,\mathcal{H}_1\right) = \frac{\left(2\pi\right)^k}{\Gamma\left(k\right)\left(2-\nu\right)^{k-1}}\left(\frac{2}{\alpha}\right)z^{-2/\alpha((2-\nu)k)-1} \times \exp\left(-\frac{2\pi}{2-\nu}z^{-2/\alpha(2-\nu)}\right)$$

Step 3: N-fold convolution

Step 3: N-fold convolution of $p\left(R_k^{-\alpha/2}\right)$

We need to derive the density of

$$\sum_{k=1}^{N_i} \sqrt{P_0} Z_k$$

We express this random sum of Y as an N-fold convolution of $Z_k, k \in \{1, \ldots, N\}$, given by

$$f_Y(y) = *_{i=1}^{N_i} f_{Z_i}(y) = \int_{-\infty}^{\infty} f_{Z_{N_i-1}}(y-w) f_{Z_{N_i}}(w) dw,$$

where $*$ represents the convolution symbol.

Each of these convolution integrals is intractable and cannot be solved analytically in closed form.

Probability Density Approximation via Series Expansion Methods

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1. We approximate the marginal likelihood via three different series expansion methods using orthogonal basis functions. These series expansions are based on a kernel density multiplied by polynomials, known as Askey polynomials:

$$f(y) = g(y)\left(1 + \sum_{j=1}^{\infty} d_j H_j(y)\right),$$

where $g(y)$ is the kernel, d_j is the j-th weight and $H_j(y)$ is the j-th order basis function.

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2. Typical Kernel densities and polynomials:
   - Gaussian density basis and Hermite polynomials.
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   - Beta density basis and Jacobi polynomials.
3. Properties of orthogonality between density functions and polynomials guarantee the integration of density to equal to one.

Some bad news

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1. These expansions do not ensure positivity of the density at all points (it can be negative for particular choices of Skew and Kurtosis).
2. It is important to characterize these values that produce the "envelope" for the density approximation in which it will remain positive. This characterization can be carried out by finding the appropriate regions in the Skew-Kurtosis plane (S-K plane) which generate positive support.

Gram-Charlier Series Expansion

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The Gram-Charlier series expansion utilises a Gaussian kernel, $g(y)$, and Hermite polynomials, $H_s(x)$, as basis function. These polynomials are defined in terms of the derivatives of the normal density, $g(y)$ as follows:

$$\frac{\mathrm{d}^s g(y)}{\mathrm{d}^s y} = (-1)^s H_s(y) g(y).$$

The Gram-Charlier series expansion is given by:

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\kappa_2} \exp\left(-\frac{(y-\kappa_1)^2}{2\kappa_2^2}\right) \sum_{r=1}^{\infty} \frac{\kappa_r}{r!} \frac{-\mathrm{d}^r (g(y))}{\mathrm{d}^r y},$$

where κ_r is the r-th cumulant of Y.

Lemma (Gram-Charlier A Series Expansion:)

The fourth order approximation of a probability distribution, $f_Y(y)$, via the Gram-Charlier A series is given by

$$f_Y(y) \approx \frac{1}{\sqrt{2\pi}\kappa_2} \exp\left(-\frac{(y-\kappa_1)^2}{2{\kappa_2}^2}\right) \times \left(1 + \frac{\kappa_3}{6{\kappa_2}^3} H_3\left(\frac{y-\kappa_1}{\kappa_2}\right) + \frac{\kappa_4}{24{\kappa_2}^4} H_4\left(\frac{y-\kappa_1}{\kappa_2}\right)\right),$$

where $H_3(y) = y^3 - 3y$ and $H_4(y) = y^4 - 6y^2 + 3$ are the Hermite polynomials, and $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ are the first, second, third and fourth cumulants of Y.

Gamma-Laguerre Series Expansion

The Gamma-Laguerre series expansion utilises the orthogonality between the Gamma density kernel and the Laguerre polynomials:

$$L_n^{(\alpha)}(x) = (-1)^n x^{1-a} \exp(-x) \frac{d^n}{dx^n} \left(x^{n+a-1} \exp(-x) \right).$$

The first five orthonormal polynomial basis are given by:

$$\begin{aligned}
L_0^{(a)}(x) &= 1 \\
L_1^{(a)}(x) &= x - a \\
L_2^{(a)}(x) &= x^2 - 2(a+1)x + (a+1)a \\
L_3^{(a)}(x) &= x^3 - 3(a+2)x^2 + 3(a+2)(a+1)x - (a+2)(a+1)a \\
L_4^{(a)}(x) &= x^4 - 4(a+3)x^3 + 6(a+3)(a+2)x^2 - 4(a+3)(a+2)(a+1)x \\
&\quad + (a+3)(a+2)(a+1)a
\end{aligned}$$

Gamma-Laguerre Series Expansion

Instead of directly working with y, we first rescale it to a R.V. $\widetilde{y}$ by $\widetilde{y} = by$, where $b = \frac{\mathbb{E}[y]}{\mathrm{Var}[y]}$ and set $a = \frac{\mathbb{E}[y]^2}{\mathrm{Var}[y]}$. Denoting the density of $\widetilde{y}$ as $f_{\widetilde{y}}$, we express $f_{\widetilde{y}}$ as follows:

$$f_{\widetilde{y}}\left(\widetilde{y}\right) = g\left(\widetilde{y}; a\right) \sum_{n=1}^{\infty} A_n L_n^{(a)}\left(\widetilde{y}\right),$$

where the kernel is the Gamma density, ie. $g\left(\widetilde{y}; a\right) = \frac{\widetilde{y}^{a-1} \exp^{-\widetilde{y}}}{\Gamma(a)}$, with shape $=$ a and scale $=$ 1.

$$\begin{aligned}
A_0 &= 1 \\
A_1 &= 0 \\
A_2 &= 0 \\
A_3 &= \frac{\Gamma\left(a\right)}{3!\Gamma(a+3)}(\mu_3 - 2a) \\
A_4 &= \frac{\Gamma\left(a\right)}{4!\Gamma(a+4)}(\mu_4 - 12\mu_3 - 3a^2 + 18a),
\end{aligned}$$

where $\mu_n = \mathbb{E}\left[(X - \mathbb{E}\left[X\right])^n\right]$.

Beta-Jacobi Series Expansion

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This new expansion is relevant for cases where Y has a bounded support $[a, b]$. To achieve this, we construct the series based on a Beta kernel and the Jacobi polynomials. It is important to note that the Jacobi polynomials are only orthogonal on $[-1, 1]$. Hence, we need to transform Y so that it has also has support $[-1, 1]$. This is achieved via the transformation $X = \frac{2}{b-a}\left(Y - \frac{a+b}{2}\right)$. The Beta-Jacobi series expansion is given by:

$$f_X(x) = \frac{(x+1)^{\theta-1}(1-x)^{\eta-1}}{B(\theta,\eta)2^{\theta+\eta-1}} \sum_{i=0}^{d} a_i P_i^{(\eta-1,\theta-1)}(x),$$

where the coefficients, a_i, and the Jacobi polynomials, $P_i^{(\eta-1,\theta-1)}(x)$, are given by:

$$a_i = \sum_{j=0}^{i} \mathbb{E}\left[X^j\right] \frac{B(\theta,\eta)(2i+\theta+\eta-1)\, i!}{\Gamma(i+\theta)} \sum_{m=j}^{i} \frac{\Gamma(\eta+\theta+i+m-1)}{\Gamma(i-m+1)\Gamma(\eta+m)\, m! 2^m} \binom{m}{j} (-1)^{m-j}$$

$$P_i^{(\eta-1,\theta-1)}(x) = \frac{\Gamma(\eta+i)}{\Gamma(\eta+\theta+i-1)} \sum_{m=0}^{i} \frac{\Gamma(\eta+\theta+i+m-1)}{\Gamma(i-m+1)\Gamma(\eta+m+1)\, m!} \left(\frac{x-1}{2}\right)^m.$$

Calculating the moments & cumulants

Lemma

The m-th moment of Z_k is given by:

1. *BPP deployment:*

$$\mathbb{E}\left[Z_k^m\right] = \begin{cases} R^{-m\alpha/2} \dfrac{\Gamma(N_B+1)\Gamma\left(k-\frac{m\alpha}{2(2-\nu)}\right)}{\Gamma(k)\Gamma\left(N_B-\frac{m\alpha}{2(2-\nu)}+1\right)}, & k-\dfrac{m\alpha}{2(2-\nu)} \notin \mathbb{Z}_{\leq 0} \\ \infty, & \textit{otherwise} \end{cases}$$

2. *PPP deployment:*

$$\mathbb{E}\left[Z_k^m\right] = \begin{cases} \left(\frac{\nu+2}{2\pi}\right)^{-\frac{\alpha m}{2(\nu+2)}} \dfrac{\Gamma\left(k-\frac{\alpha m}{2(\nu+2)}\right)}{\Gamma(k)}, & k-\dfrac{\alpha m}{2(\nu+2)} \notin \mathbb{Z}_{\leq 0} \\ \infty, & \textit{otherwise} \end{cases}$$

Positive support analysis

Series expansions positive support

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3. This characterization can be carried out by finding the appropriate regions in the Skew-Kurtosis plane (S-K plane) which generate positive support.

Gram-Charlier Series Expansion

The fourth order approximation of a probability distribution, $f_{\tilde{Y}}(\tilde{y})$, via the Gram-Charlier A series is given by

$$f_{\tilde{Y}}(\tilde{y}) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tilde{y}^2}{2}\right)\left(1 + \frac{s}{6} H_3\left(\tilde{y}\right) + \frac{\kappa_4}{24} H_4\left(\tilde{y}\right)\right)$$

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For $f_{\tilde{Y}}(\tilde{y})$ to be positive for every $\tilde{Y}$:

$$1 + \frac{s}{6} H_3(\tilde{y}) + \frac{\kappa_4}{24} H_4(\tilde{y}) \geq 0$$

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$$1 + \frac{s}{6} H_3\left(\tilde{y}\right) + \frac{\kappa_4}{24} H_4\left(\tilde{y}\right) \geq 0$$

Using notions of analytical geometry, Jondeau Et. al. obtained the following boundary conditions.

Gram-Charlier Series Expansion

Lemma (Positive density conditions :)

The Gram-Charlier A series expansion yields positive values for the density $f_Y(y)$ only if:

$$\begin{cases} s(\tilde{y}) & = -24\frac{H_3(\tilde{y})}{d(\tilde{y})}, \\ k(\tilde{y}) & = 72\frac{H_2(\tilde{y})}{d(\tilde{y})}, \end{cases}$$

where $\tilde{Y} = \frac{Y-\kappa_1}{\kappa_2}$ *and* $d(\tilde{y}) = 4H_3^2(\tilde{y}) - 3H_2(\tilde{y})H_4(\tilde{y})$

Gamma-Laguerre Series Expansion

Lemma (Positive density conditions:)

The Gamma-Laguerre series expansion yields positive values for the density $f_X(x)$ if:

$$\begin{cases} k(x) = (\frac{B_1' B_3}{B_1} - B_3')(B_2' - \frac{B_1' B_2}{B_1})^{-1} \\ s(x) = -\frac{1}{B_1}(\mu_4(x) B_2 + B_3) \end{cases}, \quad \text{for } x \in [0, +\infty)$$

where B_1, B_2, B_3 are given by:

$$B_1(x) = \frac{x^{a-1}\exp(-x)}{\Gamma(a)} \left(\frac{\Gamma(a)}{3!\Gamma(a+3)} L_3^{(a)}(x) - 12 \frac{\Gamma(a)}{4!\Gamma(a+4)} L_4^{(a)}(x) \right),$$

$$B_2(x) = \frac{x^{a-1}\exp(-x)}{\Gamma(a)} \frac{\Gamma(a)}{4!\Gamma(a+4)} L_4^{(a)}(x),$$

$$B_3(x) = \frac{x^{a-1}\exp(-x)}{\Gamma(a)} \left(1 - 2a \frac{\Gamma(a)}{3!\Gamma(a+3)} L_3^{(a)}(x) + (-3a^2 + 18a) \frac{\Gamma(a)}{4!\Gamma(a+4)} L_4^{(a)}(x) \right).$$

Beta-Jacobi Series Expansion

Theorem (Positive density conditions:)

The Beta-Jacobi series expansion yields positive values for the density $f_X(x)$ if:

$$\begin{cases} k(x) & = \left(\frac{B_1' B_3}{B_1} - B_3'\right)\left(B_2' - \frac{B_1' B_2}{B_1}\right)^{-1} \\ s(x) & = -\frac{1}{B_1}\left(\mu_4(x) B_2 + B_3\right) \end{cases}, \text{for } x \in [-1, +1)$$

where

$$B_1 = \left(C_{33} P_3 + C_{43} P_4\right) \frac{(x+1)^{\theta-1}(1-x)^{\eta-1}}{B(\theta,\eta) 2^{\theta+\eta-1}},$$

$$B_2 = C_{44} P_4 \frac{(x+1)^{\theta-1}(1-x)^{\eta-1}}{B(\theta,\eta) 2^{\theta+\eta-1}},$$

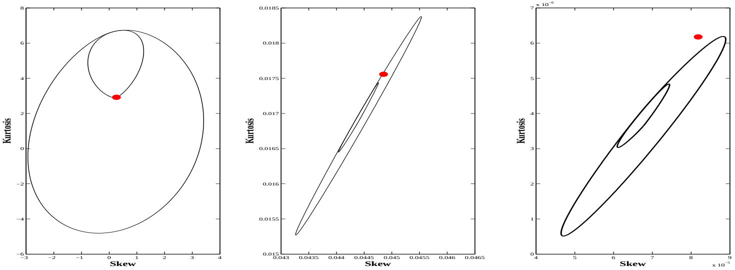
$$B_3 = \frac{(x+1)^{\theta-1}(1-x)^{\eta-1}}{B(\theta,\eta) 2^{\theta+\eta-1}} \left(\sum_{i=0}^{2} a_i P_i + \left(C_{30} + C_{31}\mu_1 + C_{32}\mu_2\right) P_3 + \left(C_{40} + C_{41}\mu_1 + C\right.\right.$$

$$C_{ij} = \frac{B(\theta,\eta)(2i+\theta+\eta-1)\, i!}{\Gamma(i+\theta)} \sum_{m=j}^{i} \frac{\Gamma(\eta+\theta+i+m-1)}{\Gamma(i-m+1)\Gamma(\eta+m)\, m! 2^m} \binom{m}{j} (-1)^{m-j},$$

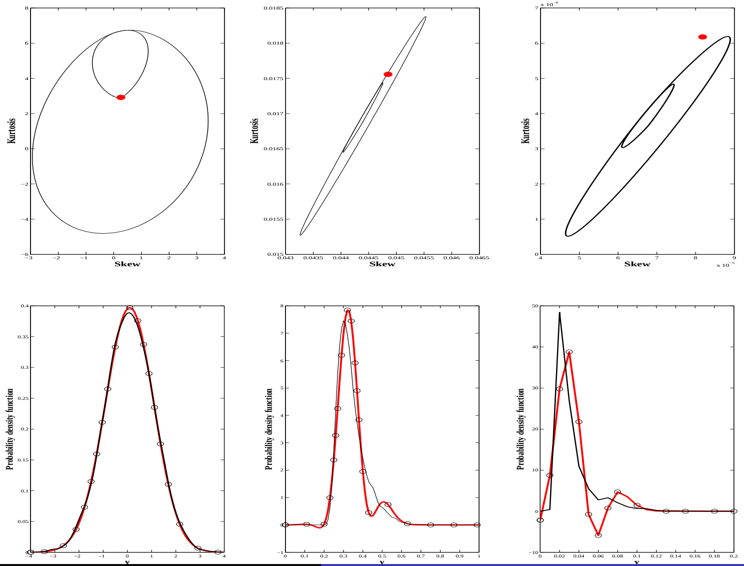
and $\mu_1 = \mathbb{E}[X], \mu_2 = \mathbb{E}\left[X^2\right]$.

Simulations

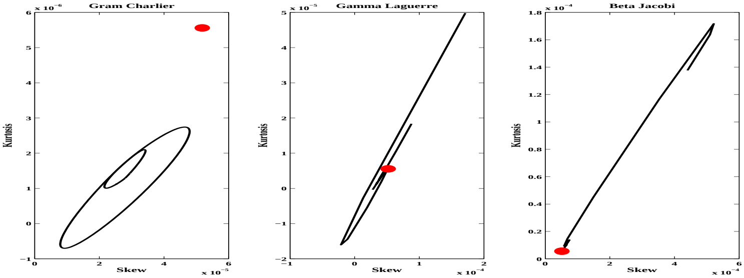
Gram-Charlier series expansion



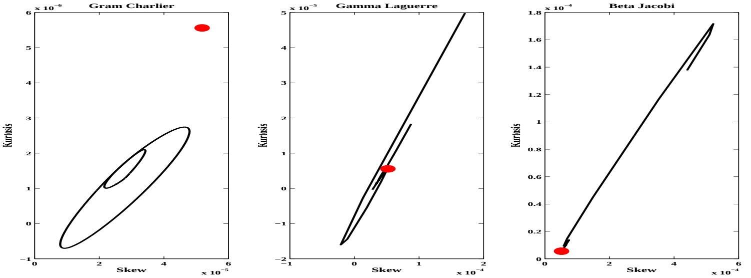
Gram-Charlier series expansion

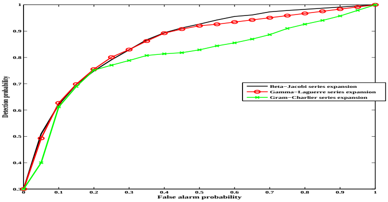


Receiver Operational Characteristics

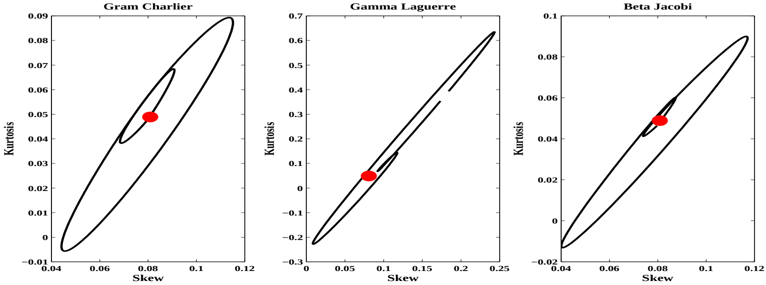


Receiver Operational Characteristics

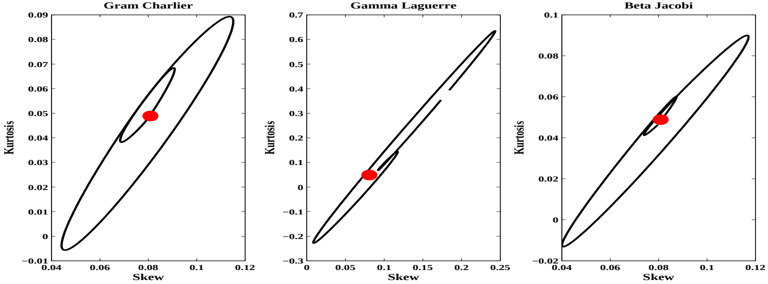


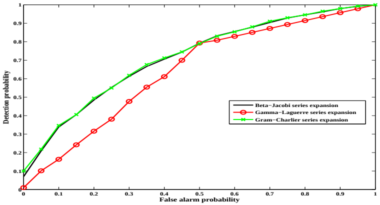


Receiver Operational Characteristics



Receiver Operational Characteristics





Conclusions

1. Addressed the problem of event detection in sensor networks with random deployments.
2. Developed three different analytic approximations via series expansions of the marginal likelihood.
3. Analysed the regions of positive suuport under each expansion.

Questions?

Thanks very much!

Questions?