Estimation of Spatially Correlated Random Fields in Heterogeneous Wireless Sensor Networks

Ido Nevat

Sense & Sense-Abilities (S&S)
I2R
A*STAR

Joint work with Gareth Peters (UCL), Francois Spetier (Telecom1 Lille) and Tomoko Matsui (ISM)

July 29, 2014

Outline

1. Introduction to random processes
2. Wireless sensor network system model
3. Estimation goals and criteria
4. Algorithms development
5. Simulations
6. Conclusions

Stochastic Processes and Random Fields

Definition (Stochastic process)

Given a parameter space X, a stochastic process f over X is a collection of random variables

$$\{f(\mathbf{x}) : \mathbf{x} \in X\}.$$

Definition (Stochastic process)

Given a parameter space X, a stochastic process f over X is a collection of random variables

$$\{f(\mathbf{x}) : \mathbf{x} \in X\}.$$

Definition (Gaussian Random Field)

A random field f on a parameter set X for which the (finite dimensional) distributions of $(f(\mathbf{x}_1), \cdots, f(\mathbf{x}_k))$ are multivarite Gaussian for each $1 \leq k \leq \infty$ and each $(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in X^k$.

Stochastic Processes and Random Fields

Definition (Stochastic process)

Given a parameter space X, a stochastic process f over X is a collection of random variables

$$\{f(\mathbf{x}) : \mathbf{x} \in X\}.$$

Definition (Gaussian Random Field)

A random field f on a parameter set X for which the (finite dimensional) distributions of $(f(\mathbf{x}_1), \cdots, f(\mathbf{x}_k))$ are multivarite Gaussian for each $1 \leq k \leq \infty$ and each $(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in X^k$.

Gaussian random fields are determined by their *mean* and *covariance* functions:

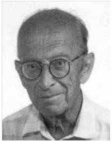
$$\mu(\cdot;\boldsymbol{\theta}) \triangleq \mathbb{E}[f(\cdot)] : \mathbb{R}^n \mapsto \mathbb{R}$$

$$\mathcal{C}(\cdot,\cdot;\boldsymbol{\Psi}) \triangleq \mathbb{E}[(f(\cdot) - \mu(\cdot;\boldsymbol{\theta}))(f(\cdot) - \mu(\cdot;\boldsymbol{\theta}))] : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$$

The covariance function is a measure of similarity and smoothness of the random field.

Some common covariance functions:

1. Linear: $\mathcal{C}\left(\mathbf{x}_1, \mathbf{x}_2; \mathbf{\Psi}\right) = \mathbf{x}_1^T \mathbf{x}_2$
2. Exponential: $\mathcal{C}\left(\mathbf{x}_1, \mathbf{x}_2; \mathbf{\Psi}\right) = \exp\left(-\left(\frac{|\mathbf{x}_2 - \mathbf{x}_1|}{\theta_1}\right)^{\theta_2}\right)$
3. Matérn:
$\mathcal{C}\left(\mathbf{x}_1, \mathbf{x}_2; \mathbf{\Psi}\right) = \frac{2^{1-\nu}}{\Gamma(\nu)}\left(\frac{\sqrt{2\nu}|\mathbf{x}_2 - \mathbf{x}_1|}{l}\right)^{\nu} \mathcal{K}_{\nu}\left(\frac{\sqrt{2\nu}|\mathbf{x}_2 - \mathbf{x}_1|}{l}\right)$

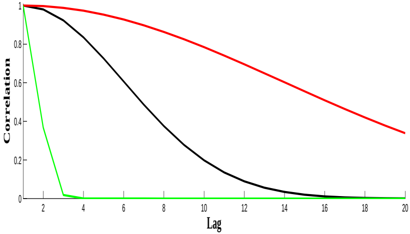


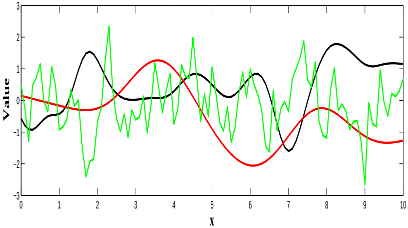
Bertil Matérn

James Mercer

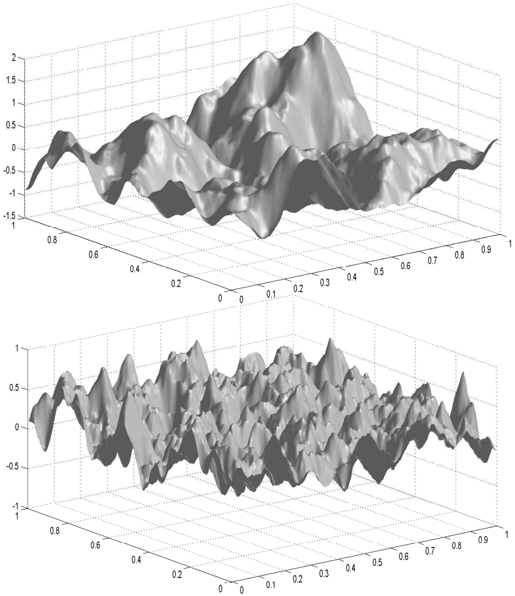
Example: Gaussian Processes with exponential kernel

$$\mathcal{C}\left(\mathbf{x}_1, \mathbf{x}_2; \mathbf{\Psi}\right) = \exp\left(-\left(\frac{|\mathbf{x}_2 - \mathbf{x}_1|}{\theta_1}\right)^{\theta_2}\right)$$





Example: 2-D Gaussian Processes



Why Gaussian ?

A few good reasons for using Gaussian Random Fields:

- Good approximation for many physical phenomena found in nature (ecology, geology, epidemiology, geography, image analysis, meteorology, forestry, geosciences....)

Why Gaussian ?

A few good reasons for using Gaussian Random Fields:

- Good approximation for many physical phenomena found in nature (ecology, geology, epidemiology, geography, image analysis, meteorology, forestry, geosciences....)
- Fully characterized with two moments

Why Gaussian ?

A few good reasons for using Gaussian Random Fields:

- Good approximation for many physical phenomena found in nature (ecology, geology, epidemiology, geography, image analysis, meteorology, forestry, geosciences....)
- Fully characterized with two moments
- Likelihood accessible (conjugate model)

Why Gaussian ?

A few good reasons for using Gaussian Random Fields:

- Good approximation for many physical phenomena found in nature (ecology, geology, epidemiology, geography, image analysis, meteorology, forestry, geosciences....)
- Fully characterized with two moments
- Likelihood accessible (conjugate model)
- Conditional expectation is linear

Why Gaussian ?

A few good reasons for using Gaussian Random Fields:

- Good approximation for many physical phenomena found in nature (ecology, geology, epidemiology, geography, image analysis, meteorology, forestry, geosciences....)
- Fully characterized with two moments
- Likelihood accessible (conjugate model)
- Conditional expectation is linear
- Stability under linear combinations, marginalization and conditioning

The problem

The Random Field Reconstruction problem:

Given observations from sensors which are deployed in the field, to perform estimation regarding some attributes of the field at un-monitored locations.

The problem

The Random Field Reconstruction problem:

> Given observations from sensors which are deployed in the field, to perform estimation regarding some attributes of the field at un-monitored locations.

If the observations are "Analog" (linear transformation of the intensity of the field + additive Gaussian noise), inference via Gaussian Process regression is trivial to perform.

The problem

The Random Field Reconstruction problem:

> Given observations from sensors which are deployed in the field, to perform estimation regarding some attributes of the field at un-monitored locations.

If the observations are "Analog" (linear transformation of the intensity of the field + additive Gaussian noise), inference via Gaussian Process regression is trivial to perform.
In many cases, it's impossible to place "Analog" sensors in locations of interest, due to transmission power constraint etc.

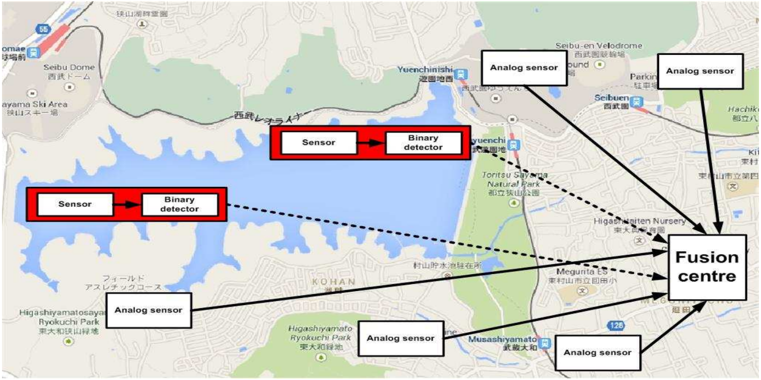
The problem

The Random Field Reconstruction problem:

Given observations from sensors which are deployed in the field, to perform estimation regarding some attributes of the field at un-monitored locations.

If the observations are “Analog” (linear transformation of the intensity of the field + additive Gaussian noise), inference via Gaussian Process regression is trivial to perform.
In many cases, it’s impossible to place “Analog” sensors in locations of interest, due to transmission power constraint etc.
Instead, it is possible to place “Digital” sensors in problematic locations.

Heterogeneous sensor network deployment



Heterogeneous sensor network deployment

Our goal is to develop a new approach to fuse mixed analog/digital observations in order to perform spatial field reconstruction.

System Model

A1 A random spatial phenomenon defined over a 2-dimensional space $\mathcal{X} \in \mathbb{R}^2$. The mean of the physical process is modelled by a smooth continuous spatial function $\mathbf{f}(\cdot) : \mathcal{X} \mapsto \mathbb{R}$, modelled *a-priori* as a Gaussian Process:

$$\mathcal{F} := \left\{ f(\cdot) : \mathbb{R}^2 \mapsto \mathbb{R} \text{ s.t. } f(\cdot) \sim \mathcal{GP}\left(\mu\left(\cdot;\boldsymbol{\theta}\right), \mathcal{C}\left(\cdot,\cdot;\boldsymbol{\Psi}\right)\right), \right.$$
$$\left. \text{with } \mu\left(\cdot;\boldsymbol{\theta}\right) : \mathbb{R}^2 \mapsto \mathbb{R}, \text{ and } \mathcal{C}\left(\cdot,\cdot;\boldsymbol{\Psi}\right) : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R} \right\}.$$

System Model

A1 A random spatial phenomenon defined over a 2-dimensional space $\mathcal{X} \in \mathbb{R}^2$. The mean of the physical process is modelled by a smooth continuous spatial function $\mathbf{f}(\cdot) : \mathcal{X} \mapsto \mathbb{R}$, modelled *a-priori* as a Gaussian Process:

$$\mathcal{F} := \left\{ f(\cdot) : \mathbb{R}^2 \mapsto \mathbb{R} \text{ s.t. } f(\cdot) \sim \mathcal{GP}\left(\mu\left(\cdot;\boldsymbol{\theta}\right), \mathcal{C}\left(\cdot,\cdot;\boldsymbol{\Psi}\right)\right), \right.$$
$$\left. \text{with } \mu\left(\cdot;\boldsymbol{\theta}\right) : \mathbb{R}^2 \mapsto \mathbb{R}, \text{ and } \mathcal{C}\left(\cdot,\cdot;\boldsymbol{\Psi}\right) : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R} \right\}.$$

A2 Let N be the number of sensors that are deployed over a 2-D region $\mathcal{X} \subseteq \mathbb{R}^2$, with $\mathbf{x}_n \in \mathcal{X}, n = \{1, \cdots, N\}$, the physical location of the n-th sensor, assumed known by the FC. The number of analog and digital sensors is N_{A} and N_{D}, respectively, so that $N = N_{\mathrm{A}} + N_{\mathrm{D}}$.

System Model

A1 A random spatial phenomenon defined over a 2-dimensional space $\mathcal{X} \in \mathbb{R}^2$. The mean of the physical process is modelled by a smooth continuous spatial function $\mathbf{f}(\cdot) : \mathcal{X} \mapsto \mathbb{R}$, modelled *a-priori* as a Gaussian Process:

$$\mathcal{F} := \{ f(\cdot) : \mathbb{R}^2 \mapsto \mathbb{R} \text{ s.t. } f(\cdot) \sim \mathcal{GP}(\mu(\cdot;\boldsymbol{\theta}), \mathcal{C}(\cdot,\cdot;\boldsymbol{\Psi})), \\ \text{with } \mu(\cdot;\boldsymbol{\theta}) : \mathbb{R}^2 \mapsto \mathbb{R}, \text{ and } \mathcal{C}(\cdot,\cdot;\boldsymbol{\Psi}) : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R} \}.$$

A2 Let N be the number of sensors that are deployed over a 2-D region $\mathcal{X} \subseteq \mathbb{R}^2$, with $\mathbf{x}_n \in \mathcal{X}, n = \{1, \cdots, N\}$, the physical location of the n-th sensor, assumed known by the FC. The number of analog and digital sensors is N_{A} and N_{D}, respectively, so that $N = N_{\text{A}} + N_{\text{D}}$.

A3 **Sensors measurement model:** each sensor collects a noisy observation of the spatial phenomenon $f(\cdot)$. At the n-th sensor, the observation is expressed as:

$$Z(\mathbf{x}_n) = f(\mathbf{x}_n) + W_n, \; n = \{1, \cdots, N\}$$

where W_n is i.i.d Gaussian noise $W_n \sim N(0, \sigma_{\text{W}}^2)$.

System Model

A4 **Analog sensors processing:** each of the N_A analog sensors transmits its noisy observation to the FC over AWGN channels:

$$Y_n^A = Z(\mathbf{x}_n) + V_n, \quad n = \{1, \ldots, N_\text{A}\},$$

where V_n is i.i.d Gaussian noise $V_n \sim N(0, \sigma_\text{V}^2)$.

System Model

A4 **Analog sensors processing:** each of the N_A analog sensors transmits its noisy observation to the FC over AWGN channels:

$$Y_n^A = Z(\mathbf{x}_n) + V_n, \quad n = \{1, \dots, N_A\},$$

where V_n is i.i.d Gaussian noise $V_n \sim N(0, \sigma_V^2)$.

A5 **Digital Sensors processing:**

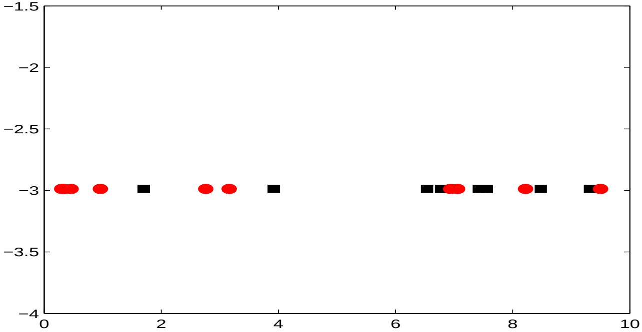
1. **Thresholding:** at the n-th digital sensor, $n = \{1, \dots, N_D\}$, a thresholding process is given as follows:

$$Z(\mathbf{x}_n) \underset{B(\mathbf{x}_n) = -1}{\overset{B(\mathbf{x}_n)=1}{\gtrless}} \lambda,$$

   where λ is a pre-defined threshold.

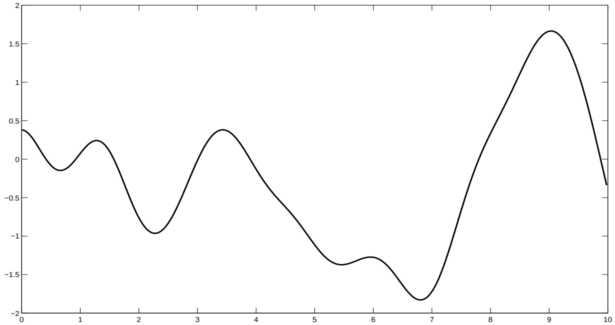
2. **Wireless Communications to Fusion Centre Model:** the decision $B(\mathbf{x}_n)$ is transmitted to the FC over imperfect binary wireless channels, with transition probabilities $p_{0,0}, p_{0,1}, p_{1,0}, p_{1,1}$.

Example



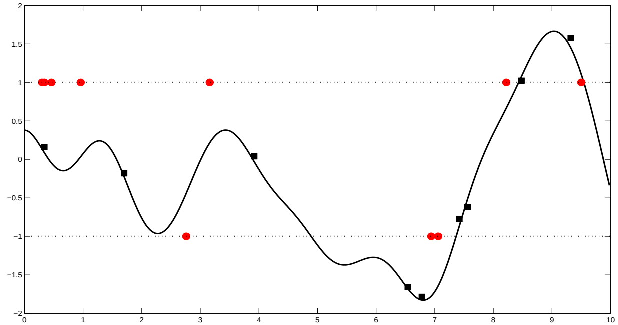
Sensors deployment: black - analog sensors, red - digital sensors

Example



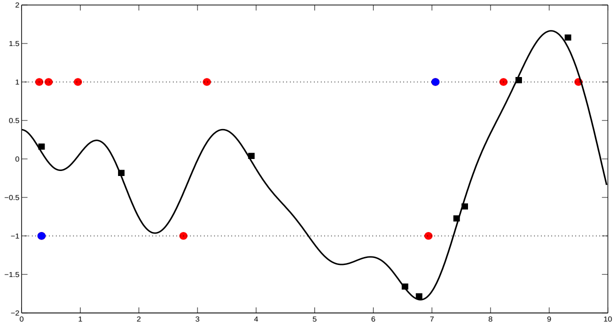
Realisation from a 1-D Gaussian Process

Example



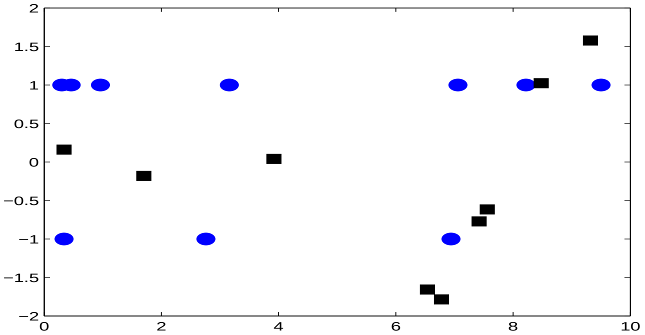
Noisy observations of Analog and Digital sensors

Example



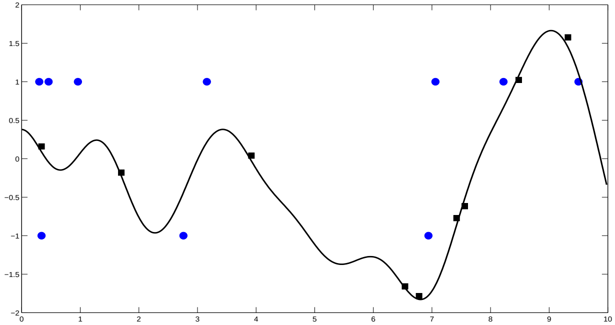
Wireless channel effects

Example



Analog and Digital observations

Example



Field reconstruction

Estimation Objectives

Estimation Objectives

1 **Objective I: MMSE spatial random field reconstruction-** Accurately reconstruct (i.e. estimate) the spatial random field at un-monitored locations, $\mathbf{x}_* \in \Omega$, from samples collected by the N sensors . The Minimum Mean Squared Error (MMSE) utilises the following distortion metric:

$$D\left(\widehat{f}_*, f_*\right) := \mathbb{E}\left[\left(f_* - \widehat{f}_*\right)^2\right].$$

The optimal solution in the sense of minimising this distortion metric is the posterior predictive mean, given by the solution to the following integral:

$$\widehat{f}_* = \mathbb{E}\left[f_* | \mathbf{x}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{Y}_{\mathcal{N}}\right] = \int f_* p\left(f_* | \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) \mathrm{d}f_*.$$

Estimation Objectives

2 **Objective II: spatial exeedance map-**
Construct a spatial exceedance map estimation is quantified by the following metric: find a region $D_e \subset \Omega$ such that, with a certain given probability, $f(\mathbf{x}) \geq T$ for all $\mathbf{x} \in D_e$ for a given level T:

$$\begin{aligned} D_e &= \{\mathbf{x} : \mathbb{P}(f_* \geq T | \mathbf{x}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{Y}_{\mathcal{N}}) \geq 1 - \alpha\} \\ &= \left\{\mathbf{x} : \int_T^{\infty} p(f_* | \mathbf{x}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{Y}_{\mathcal{N}}) \, \mathrm{d}f_* \geq 1 - \alpha\right\}, \end{aligned}$$

where T is a pre-defined threshold and α is the confidence level and D_e is the domain or set of $\mathbf{x}$ values satisfying the exceedance of the spatial field.

Estimation Objectives

3 **Objective III: Spatial Classification-**
Predict the confidence for each class at un-monitored locations, $\mathbf{x}_* \in \Omega$. That means that we find the classifier $\widehat{B}_* : \Omega \leftrightarrow \{0, 1\}$ that minimizes the error probability $\mathbb{P}\left(B_* \neq \widehat{B}_*\right)$, at an arbitrary location $\mathbf{x}_* \in \mathcal{X}$.
This requires the calculation of the binary conditional predictive distribution in closed form, given by:

$$\mathbb{P}\left(B_* = 0|\mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}, \lambda\right) = \int \mathbb{P}\left(B_* = 0|f_*, \mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}, \lambda\right) p\left(f_*|\mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right) \mathrm{d}f_*,$$

$$\mathbb{P}\left(B_* = 1|\mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}, \lambda\right) = \int \mathbb{P}\left(B_* = 1|f_*, \mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}, \lambda\right) p\left(f_*|\mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right) \mathrm{d}f_*.$$

and the classifier

$$\widehat{B}_* = \begin{cases} 1 & , \mathbb{P}\left(B_*|\mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right) \geq \lambda \\ 0 & , \mathbb{P}\left(B_*|\mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right) < \lambda. \end{cases}$$

Estimation Objectives

The common feature of Objectives $1-3$ is the posterior predictive distribution $p\left(f_*|\mathbf{x}_*, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right)$.

Estimation Objectives

The common feature of Objectives $1 - 3$ is the posterior predictive distribution $p\left(f_* | \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right)$.

$$
\begin{aligned}
p\left(f_* | \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) &= \int \ldots \int_{\mathbb{R}^{\mathcal{N}}} p\left(f_* | \mathbf{f}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{N}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) d\mathbf{f}_{\mathcal{N}} \\
&= \int \ldots \int_{\mathbb{R}^{\mathcal{N}}} p\left(f_* | \mathbf{f}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{A}} | \mathbf{f}_{\mathcal{D}}, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) d\mathbf{f}_{\mathcal{N}}
\end{aligned}
$$

1. $p\left(f_* | \mathbf{f}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}\right)$: conditional predictive prior distribution.
2. $p\left(\mathbf{f}_{\mathcal{A}} | \mathbf{f}_{\mathcal{D}}, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right)$: posterior distribution for the spatial phenomenon at the analog sensor locations given observations.
3. $p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right)$: posterior distribution for the spatial phenomenon at the digital sensor locations given observations.

$$p\left(f_*|\mathbf{x}_*,\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) = \int \ldots \int_{\mathbb{R}^\mathcal{N}} p\left(f_*|\mathbf{f}_\mathcal{N},\mathbf{x}_*,\mathbf{x}_\mathcal{N}\right) p\left(\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D},\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) p\left(\mathbf{f}_\mathcal{D}|\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) d\mathbf{f}_\mathcal{N}$$

Lemma

The conditional predictive prior distribution, $p\left(f_*|\mathbf{f}_\mathcal{N},\mathbf{x}_*,\mathbf{x}_\mathcal{N}\right)$, is given by:

$$p\left(f_*|\mathbf{f}_\mathcal{N},\mathbf{x}_*,\mathbf{x}_\mathcal{N}\right) = N\left(f_*;\mu_{f_*|\mathbf{f}_\mathcal{N}},\sigma^2_{f_*|\mathbf{f}_\mathcal{N}}\right)$$

$$\mu_{f_*|\mathbf{f}_\mathcal{N}} := \mu\left(\mathbf{x}_*\right) + k\left(\mathbf{x}_*,\mathbf{x}_\mathcal{N}\right)\mathbf{K}^{-1}\left(\mathbf{x}_\mathcal{N},\mathbf{x}_\mathcal{N}\right)\left(\mathbf{f}_\mathcal{N} - \mu\left(\mathbf{x}_\mathcal{N}\right)\right)$$

$$\sigma^2_{f_*|\mathbf{f}_\mathcal{N}} := k\left(\mathbf{x}_*,\mathbf{x}_*\right) - k\left(\mathbf{x}_*,\mathbf{x}_\mathcal{N}\right)\mathbf{K}^{-1}\left(\mathbf{x}_\mathcal{N},\mathbf{x}_\mathcal{N}\right)k\left(\mathbf{x}_\mathcal{N},\mathbf{x}_*\right)$$

$$p\left(f_* | \mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right) = \int \ldots \int_{\mathbb{R}^\mathcal{N}} p\left(f_* | \mathbf{f}_\mathcal{N}, \mathbf{x}_*, \mathbf{x}_\mathcal{N}\right) p\left(\mathbf{f}_\mathcal{A} | \mathbf{f}_\mathcal{D}, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right) p\left(\mathbf{f}_\mathcal{D} | \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right) d\mathbf{f}_\mathcal{N}$$

Lemma

The conditional distribution, $p\left(\mathbf{f}_\mathcal{A} | \mathbf{f}_\mathcal{D}, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right)$, *is given by:*

$$p\left(\mathbf{f}_\mathcal{A} | \mathbf{f}_\mathcal{D}, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right) = N\left(\mathbf{f}_\mathcal{A}; \boldsymbol{\mu}_{\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D}, \mathbf{Y}_\mathcal{N}}, \boldsymbol{\Sigma}_{\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D}, \mathbf{Y}_\mathcal{N}}\right)$$

$$\boldsymbol{\mu}_{\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D}, \mathbf{Y}_\mathcal{N}} := \left(\boldsymbol{\Sigma}^{-1}_{\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D}} + \sigma_W^{-2}\mathbf{I}\right)^{-1} \left(\boldsymbol{\Sigma}^{-1}_{\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D}} \boldsymbol{\mu}_{\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D}} + \sigma_W^{-2}\mathbf{Y}_\mathcal{A}\right)$$

$$\boldsymbol{\Sigma}_{\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D}, \mathbf{Y}_\mathcal{N}} := \left(\boldsymbol{\Sigma}^{-1}_{\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D}} + \sigma_W^{-2}\mathbf{I}\right)^{-1}.$$

The posterior distribution

$$p\left(f_*|\mathbf{x}_*,\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) = \int \ldots \int_{\mathbb{R}^\mathcal{N}} p\left(f_*|\mathbf{f}_\mathcal{N},\mathbf{x}_*,\mathbf{x}_\mathcal{N}\right) p\left(\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D},\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) p\left(\mathbf{f}_\mathcal{D}|\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) d\mathbf{f}_\mathcal{N}$$

Using Bayes' law, the posterior distribution for the spatial phenomenon at the digital sensor locations is given by

$$p\left(\mathbf{f}_\mathcal{D}|\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) =$$

The posterior distribution

$$p\left(f_*|\mathbf{x}_*,\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) = \int\ldots\int_{\mathbb{R}^\mathcal{N}} p\left(f_*|\mathbf{f}_\mathcal{N},\mathbf{x}_*,\mathbf{x}_\mathcal{N}\right) p\left(\mathbf{f}_\mathcal{A}|\mathbf{f}_\mathcal{D},\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) p\left(\mathbf{f}_\mathcal{D}|\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) d\mathbf{f}_\mathcal{N}$$

Using Bayes' law, the posterior distribution for the spatial phenomenon at the digital sensor locations is given by

$$p\left(\mathbf{f}_\mathcal{D}|\mathbf{x}_\mathcal{N},\mathbf{Y}_\mathcal{N}\right) = \frac{\mathbb{P}\left(\mathbf{Y}_\mathcal{N}|\mathbf{x}_\mathcal{N},\mathbf{f}_\mathcal{D}\right) p\left(\mathbf{f}_\mathcal{D}|\mathbf{x}_\mathcal{N}\right)}{\mathbb{P}\left(\mathbf{Y}_\mathcal{N}|\mathbf{x}_\mathcal{N}\right)}$$

The posterior distribution

$$p\left(f_*|\mathbf{x}_*,\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}\right)=\int \ldots \int_{\mathbb{R}^{\mathcal{N}}} p\left(f_*|\mathbf{f}_{\mathcal{N}},\mathbf{x}_*,\mathbf{x}_{\mathcal{N}}\right)p\left(\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}},\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}\right)p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}\right)d\mathbf{f}_{\mathcal{N}}$$

Using Bayes' law, the posterior distribution for the spatial phenomenon at the digital sensor locations is given by

$$\begin{aligned}
p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}\right) &= \frac{\mathbb{P}\left(\mathbf{Y}_{\mathcal{N}}|\mathbf{x}_{\mathcal{N}},\mathbf{f}_{\mathcal{D}}\right)p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}\right)}{\mathbb{P}\left(\mathbf{Y}_{\mathcal{N}}|\mathbf{x}_{\mathcal{N}}\right)} \\
&= \frac{\mathbb{P}\left(\mathbf{Y}_{\mathcal{N}}|\mathbf{x}_{\mathcal{N}},\mathbf{f}_{\mathcal{D}}\right)p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}\right)}{\int \ldots \int_{\mathbb{R}^N}\mathbb{P}\left(\mathbf{Y}_{\mathcal{N}}|\mathbf{x}_{\mathcal{N}},\mathbf{f}_{\mathcal{D}}\right)p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}\right)d\mathbf{f}_{\mathcal{D}}}
\end{aligned}$$

The numerator can be easily evaluated.
However, the denominator cannot be evaluated pointwise.

Saddle Point Approximations for the predictive distribution

We approximate $p(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}})$ using using a series expansion of the Saddle-point (Laplace) type via a Gaussian basis.

Saddle Point Approximations for the predictive distribution

We approximate $p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right)$ using using a series expansion of the Saddle-point (Laplace) type via a Gaussian basis.
This transforms the intractable multiple integrals to produce simple closed form expressions. Based on these expressions we derive new algorithms and provide closed form solutions.

Saddle Point Approximations for the predictive distribution

The series expansion becomes:

$$p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right)$$

Saddle Point Approximations for the predictive distribution

The series expansion becomes:

$$p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}\right)$$
$$= \exp^{\,log(p(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}))}$$

The series expansion becomes:

$$
\begin{aligned}
& p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) \\
& = \exp^{\, log(p(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}))} \\
& = \exp^{g\left(\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}\right)+\frac{1}{2}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}\right)^{T}\nabla^{2} g|_{\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}\right)} \exp^{\widetilde{R}_{3}(\mathbf{f}_{\mathcal{D}})}
\end{aligned}
$$

The series expansion becomes:

$$
\begin{aligned}
& p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) \\
& = \exp^{\ log\left(p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right)\right)} \\
& = \exp^{g\left(\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}\right)+\frac{1}{2}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}\right)^{T} \nabla^{2} g|_{\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}\right)} \exp^{\widetilde{R}_{3}\left(\mathbf{f}_{\mathcal{D}}\right)} \\
& = \frac{1}{(2 \pi)^{N}|H|^{1 / 2}} \exp^{-\frac{1}{2}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}\right)^{T} H^{-1}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}\right)} \\
& \times \exp^{\left(g\left(\widehat{\mathbf{f}}_{\mathcal{D}}^{\mathrm{MAP}}\right)+\widetilde{R}_{3}\left(\mathbf{f}_{\mathcal{D}}\right)+\log\left((2 \pi)^{N}|H|^{1 / 2}\right)\right)}
\end{aligned}
$$

The series expansion becomes:

$$
\begin{aligned}
& p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) \\
& = \exp^{\, log(p(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}))} \\
& = \exp^{g\left(\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right) + \frac{1}{2}\left(\mathbf{f}_{\mathcal{D}} - \widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)^{T} \nabla^2 g|_{\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}} \left(\mathbf{f}_{\mathcal{D}} - \widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)} \exp^{\widetilde{R}_3(\mathbf{f}_{\mathcal{D}})} \\
& = \frac{1}{(2\pi)^{N} |H|^{1/2}} \exp^{-\frac{1}{2}\left(\mathbf{f}_{\mathcal{D}} - \widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)^{T} H^{-1} \left(\mathbf{f}_{\mathcal{D}} - \widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)} \\
& \times \exp^{\left(g\left(\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right) + \widetilde{R}_3(\mathbf{f}_{\mathcal{D}}) + \log\left((2\pi)^{N} |H|^{1/2}\right)\right)}
\end{aligned}
$$

where $H^{-1} := -\nabla^2 g|_{\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}}$.

Saddle Point Approximations for the predictive distribution

The series expansion becomes:

$$
\begin{aligned}
& p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) \\
& = \exp^{\,log(p(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}))} \\
& = \exp^{g\left(\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)+\frac{1}{2}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)^{T}\nabla^{2} g|_{\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)} \exp^{\widetilde{R}_{3}(\mathbf{f}_{\mathcal{D}})} \\
& = \frac{1}{(2\pi)^{N}|H|^{1/2}} \exp^{-\frac{1}{2}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)^{T} H^{-1}\left(\mathbf{f}_{\mathcal{D}}-\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)} \\
& \times \exp^{\left(g\left(\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)+\widetilde{R}_{3}(\mathbf{f}_{\mathcal{D}})+\log\left((2\pi)^{N}|H|^{1/2}\right)\right)}
\end{aligned}
$$

where $H^{-1} := -\nabla^2 g|_{\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}}$.
We obtain that the posterior distribution can be expressed as

$$
p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) = N\left(\mathbf{f}_{\mathcal{D}}; \widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}, H\right) \exp^{\left(g\left(\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}\right)+\widetilde{R}_{3}(\mathbf{f}_{\mathcal{D}})+\log\left((2\pi)^{N}|H|^{1/2}\right)\right)}
$$

Saddle Point Approximations for the predictive distribution

Theorem

The posterior distribution at the digital sensors, $p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right)$*:*

$$\log p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) = \log q\left(\mathbf{f}_{\mathcal{D}}; \widehat{\mathbf{f}}_{\mathcal{D}}^{MAP}, H\right) + R_3\left(\mathbf{f}_{\mathcal{D}}\right).$$

where

$$\begin{aligned} q\left(\mathbf{f}_{\mathcal{D}}; \widehat{\mathbf{f}}_{\mathcal{D}}^{MAP}, H\right) &= N\left(\mathbf{f}_{\mathcal{D}}; \widehat{\mathbf{f}}_{\mathcal{D}}^{MAP}, H^{-1}\right), \\ \widehat{\mathbf{f}}_{\mathcal{D}}^{MAP} &= \arg\max_{\mathbf{f}_{\mathcal{D}}} p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right), \\ [H]_{i,j} &= -\frac{\partial^2}{\partial f_i \partial f_j} p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right)|_{\widehat{\mathbf{f}}_{\mathcal{D}}^{MAP}}, \\ R_3\left(\mathbf{f}_{\mathcal{D}}\right) &= g\left(\widehat{\mathbf{f}}_{\mathcal{D}}^{MAP}\right) + \widetilde{R}_3\left(\mathbf{f}_{\mathcal{D}}\right) + \log\left((2\pi)^n |H|^{1/2}\right) \end{aligned}$$

The MAP estimate is given by:

$$
\begin{aligned}
\widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}} &= \arg\max_{\mathbf{f}_{\mathcal{D}}} p\left(\mathbf{f}_{\mathcal{D}} | \mathbf{Y}_{\mathcal{N}}, \mathbf{x}_{\mathcal{N}}\right) \\
&= \arg\max_{\mathbf{f}_{\mathcal{D}}} \mathbb{P}\left(\mathbf{Y}_{\mathcal{N}} | \mathbf{f}_{\mathcal{D}}, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{D}}\right) \\
&= \arg\max_{\mathbf{f}_{\mathcal{D}}} \mathbb{P}\left(\mathbf{Y}_{\mathcal{D}} | \mathbf{f}_{\mathcal{D}}, \mathbf{Y}_{\mathcal{A}}, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{Y}_{\mathcal{A}} | \mathbf{f}_{\mathcal{D}}, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{D}}\right) \\
&= \arg\max_{\mathbf{f}_{\mathcal{D}}} \Bigg(\sum_{n=1}^{N_{\text{D}}} \log \left(\sum_{l=0}^{1} \mathbb{P}\left(Y_n^D | B_n = l\right) \mathbb{P}\left(B_n = l | f_n\right) \right) \\
&\qquad + \log N\left(\mathbf{Y}_{\mathcal{A}}; \boldsymbol{\mu}_{\mathbf{f}_{\mathcal{A}} | \mathbf{f}_{\mathcal{D}}}, \left(\sigma_{\text{V}}^2 + \sigma_{\text{W}}^2\right) \mathbf{I}_{N_{\text{A}}} + \boldsymbol{\Sigma}_{\mathbf{f}_{\mathcal{A}} | \mathbf{f}_{\mathcal{D}}}\right) \\
&\qquad + \log N\left(\mathbf{f}_{\mathcal{D}}; \boldsymbol{\mu}\left(\mathbf{x}_{\mathcal{D}}\right), \mathbf{K}\left(\mathbf{x}_{\mathcal{D}}, \mathbf{x}_{\mathcal{D}}\right)\right) \Bigg).
\end{aligned}
$$

To solve this N-dimensional optimisation problem, we show that the objective function is quasi-convex and can therefore be solved exactly using any gradient based approach.
We utilse the Iterated Conditional on the Modes (ICM) of Besag to solve this problem.

Obtaining the MAP estimate

Using ICM algorithm, the MAP estimate of the n-th component of $\mathbf{f}_{\mathcal{D}}$, $\widehat{\mathbf{f}}_{\mathrm{n}}^{\mathrm{MAP}} = \arg\max_{f_n} p\left(f_n | \mathbf{x}_{\mathcal{N}}, \widehat{\mathbf{f}}_{1:\mathrm{N_D}\backslash\mathrm{n}}, \mathbf{Y}_{\mathcal{N}}\right)$, can be evaluated by solving the following one-dimensional non-linear equation:

$$\frac{\phi\left(\lambda, f\left(\mathbf{x}_n\right), \sigma_{\mathrm{W}}^2\right)\left(\mathbb{P}\left(Y_n | B_n = 0\right) - \mathbb{P}\left(Y_n | B_n = 1\right)\right)}{\mathbb{P}\left(Y_n | B_n = 1\right) + \Phi\left(\lambda, f\left(\mathbf{x}_n\right), \sigma_{\mathrm{W}}^2\right)\left(\mathbb{P}\left(Y_n | B_n = 0\right) - \mathbb{P}\left(Y_n | B_n = 1\right)\right)}$$

$$= \left(\boldsymbol{\mu}_{\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}}} - \mathbf{Y}_{\mathcal{A}}\right)^T \left(\left(\sigma_{\mathrm{V}}^2 + \sigma_{\mathrm{W}}^2\right)\mathbf{I}_{N_{\mathrm{A}}} + \boldsymbol{\Sigma}_{\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}}}\right)^{-1} K\left(\mathbf{x}_{\mathcal{A}}, \mathbf{x}_{\mathcal{D}}\right) K^{-1}\left(\mathbf{x}_{\mathcal{D}}, \mathbf{x}_{\mathcal{D}}\right)$$

$$+ \frac{\left(f\left(\mathbf{x}_n\right) - \mu_{\mathbf{x}_{\mathrm{n}}|\mathbf{f}_{\mathcal{D}}\backslash n}\right)}{\sigma^2_{\mathbf{x}_{\mathrm{n}}|\mathbf{f}_{\mathcal{D}}\backslash n}}$$

The posterior predictive distribution

Putting it all together.

The posterior predictive distribution

The posterior predictive distribution is approximated by

$$p\left(f_*|\mathbf{x}_*, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) = \int \ldots \int_{\mathbb{R}^{\mathcal{N}}} p\left(f_*|\mathbf{f}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{N}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) d\mathbf{f}_{\mathcal{N}}$$

The posterior predictive distribution

The posterior predictive distribution is approximated by

$$p\left(f_*|\mathbf{x}_*,\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}\right)=\int\ldots\int_{\mathbb{R}^{\mathcal{N}}}p\left(f_*|\mathbf{f}_{\mathcal{N}},\mathbf{x}_*,\mathbf{x}_{\mathcal{N}}\right)p\left(\mathbf{f}_{\mathcal{N}}|\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}\right)d\mathbf{f}_{\mathcal{N}}$$

$$=\int\ldots\int_{\mathbb{R}^{\mathcal{N}}}p\left(f_*|\mathbf{f}_{\mathcal{N}},\mathbf{x}_*,\mathbf{x}_{\mathcal{N}}\right)p\left(\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}},\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}\right)p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}},\mathbf{Y}_{\mathcal{N}}\right)d\mathbf{f}_{\mathcal{N}}$$

The posterior predictive distribution is approximated by

$$p\left(f_*|\mathbf{x}_*, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) = \int \ldots \int_{\mathbb{R}^{\mathcal{N}}} p\left(f_*|\mathbf{f}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{N}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) d\mathbf{f}_{\mathcal{N}}$$

$$= \int \ldots \int_{\mathbb{R}^{\mathcal{N}}} p\left(f_*|\mathbf{f}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}}, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) d\mathbf{f}_{\mathcal{N}}$$

$$\approx \int \ldots \int_{\mathbb{R}^{\mathcal{N}}} N\left(f_*; \mu_{f_*|\mathbf{f}_{\mathcal{N}}}, \sigma^2_{f_*|\mathbf{f}_{\mathcal{N}}}\right) N\left(\mathbf{f}_{\mathcal{A}}; \boldsymbol{\mu}_{\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}}, \mathbf{Y}_{\mathcal{N}}}, \boldsymbol{\Sigma}_{\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}}, \mathbf{Y}_{\mathcal{N}}}\right) N\left(\mathbf{f}_{\mathcal{D}}; \widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}, H^{-1}\right) d\mathbf{f}_{\mathcal{N}}$$

The posterior predictive distribution

The posterior predictive distribution is approximated by

$$
\begin{aligned}
& p\left(f_*|\mathbf{x}_*, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) = \int \cdots \int_{\mathbb{R}^{\mathcal{N}}} p\left(f_*|\mathbf{f}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{N}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) d\mathbf{f}_{\mathcal{N}} \\
& = \int \cdots \int_{\mathbb{R}^{\mathcal{N}}} p\left(f_*|\mathbf{f}_{\mathcal{N}}, \mathbf{x}_*, \mathbf{x}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}}, \mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) p\left(\mathbf{f}_{\mathcal{D}}|\mathbf{x}_{\mathcal{N}}, \mathbf{Y}_{\mathcal{N}}\right) d\mathbf{f}_{\mathcal{N}} \\
& \approx \int \cdots \int_{\mathbb{R}^{\mathcal{N}}} N\left(f_*; \mu_{f_*|\mathbf{f}_{\mathcal{N}}}, \sigma^2_{f_*|\mathbf{f}_{\mathcal{N}}}\right) N\left(\mathbf{f}_{\mathcal{A}}; \boldsymbol{\mu}_{\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}}, \mathbf{Y}_{\mathcal{N}}}, \boldsymbol{\Sigma}_{\mathbf{f}_{\mathcal{A}}|\mathbf{f}_{\mathcal{D}}, \mathbf{Y}_{\mathcal{N}}}\right) N\left(\mathbf{f}_{\mathcal{D}}; \widehat{\mathbf{f}}_{\mathcal{D}}^{\text{MAP}}, H^{-1}\right) d\mathbf{f}_{\mathcal{N}} \\
& = N\left(f_*; \mu_{f_*|\mathbf{Y}_{\mathcal{N}}}, \sigma^2_{f_*|\mathbf{Y}_{\mathcal{N}}}\right)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_{f_*|\mathbf{Y}_{\mathcal{N}}} &= \mu\left(\mathbf{x}_*\right) + k\left(\mathbf{x}_*, \mathbf{x}_{\mathcal{N}}\right) K^{-1}\left(\mathbf{x}_{\mathcal{N}}, \mathbf{x}_{\mathcal{N}}\right)\left(\boldsymbol{\mu}_{\mathbf{f}_{\mathcal{N}}|\mathbf{Y}_{\mathcal{N}}} - \mu\left(\mathbf{x}_{\mathcal{N}}\right)\right), \\
\sigma^2_{f_*|\mathbf{Y}_{\mathcal{N}}} &= \boldsymbol{\Sigma}^{22}_{f_*, \mathbf{f}_{\mathcal{N}}|\mathbf{Y}_{\mathcal{N}}}.
\end{aligned}
$$

Spatial field reconstruction, exceedance level estimation and spatial classification

- **Objective I: spatial MMSE random field reconstruction-**

$$\begin{aligned}\widehat{f}_* &= \mathbb{E}\left[f_* | \mathbf{x}_\mathcal{N}, \mathbf{x}_*, \mathbf{Y}_\mathcal{N}\right] \\ &\simeq \int f_* \widehat{p}\left(f_* | \mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}\right) \mathrm{d} f_* \\ &= \mu\left(\mathbf{x}_*\right) + k\left(\mathbf{x}_*, \mathbf{x}_\mathcal{N}\right) K^{-1}\left(\mathbf{x}_\mathcal{N}, \mathbf{x}_\mathcal{N}\right)\left(\boldsymbol{\mu}_{\mathbf{f}_\mathcal{N} | \mathbf{Y}_\mathcal{N}} - \mu\left(\mathbf{x}_\mathcal{N}\right)\right).\end{aligned}$$

- **Objective II: spatial exeedence map:**

$$\widehat{f}_* = \mathbb{P}\left(f_* \geq \lambda | \mathbf{x}_\mathcal{N}, \mathbf{x}_*, \mathbf{Y}_\mathcal{N}\right) \simeq 1 - \Phi\left(\lambda, \mu_{f_* | \mathbf{Y}_\mathcal{N}}, \sigma^2_{f_* | \mathbf{Y}_\mathcal{N}}\right).$$

- **Spatial Classification:**

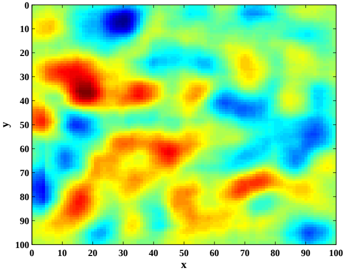
$$\mathbb{P}\left(B_* = 0 | \mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}, \lambda\right) = \Phi\left(\lambda, \mu_{f_* | \mathbf{Y}_\mathcal{N}}, \sigma^2_{\mathrm{W}} + \sigma^2_{f_* | \mathbf{Y}_\mathcal{N}}\right),$$

$$\mathbb{P}\left(B_* = 1 | \mathbf{x}_*, \mathbf{x}_\mathcal{N}, \mathbf{Y}_\mathcal{N}, \lambda\right) = 1 - \Phi\left(\lambda, \mu_{f_* | \mathbf{Y}_\mathcal{N}}, \sigma^2_{\mathrm{W}} + \sigma^2_{f_* | \mathbf{Y}_\mathcal{N}}\right).$$

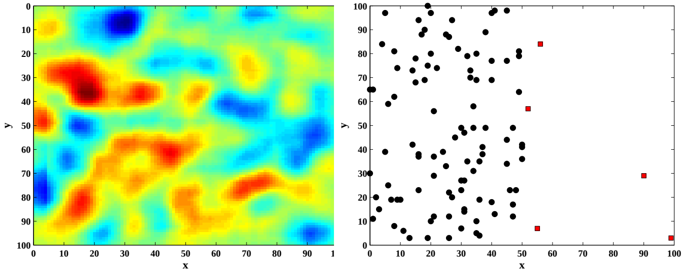
Simulations

Spatial field reconstruction

100 analog sensors and 5 digital sensors

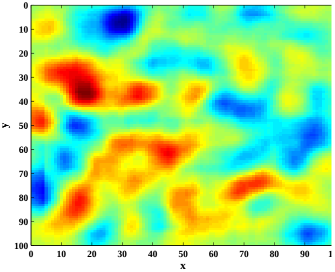


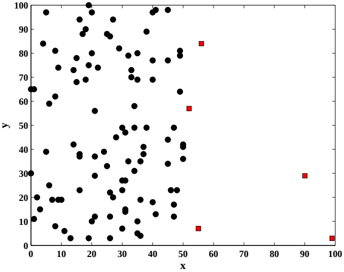
100 analog sensors and 5 digital sensors

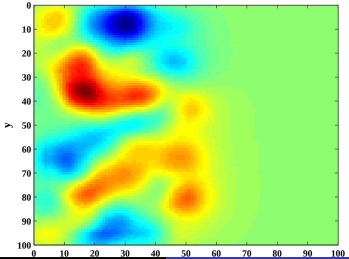


Spatial Field Reconstruction

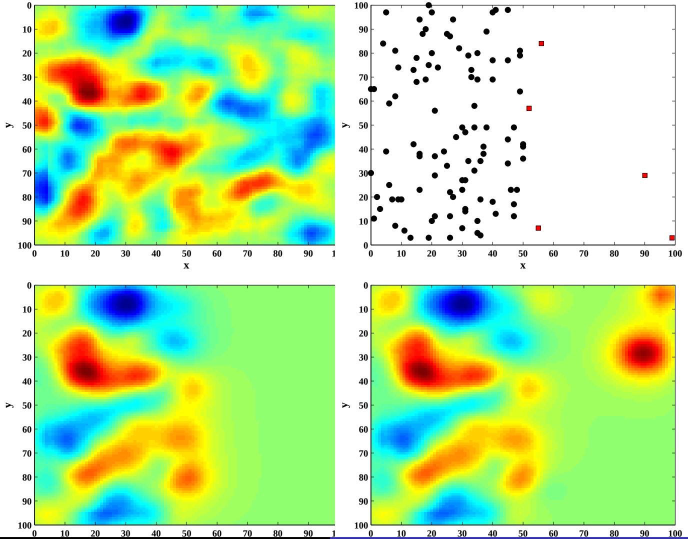
100 analog sensors and 5 digital sensors





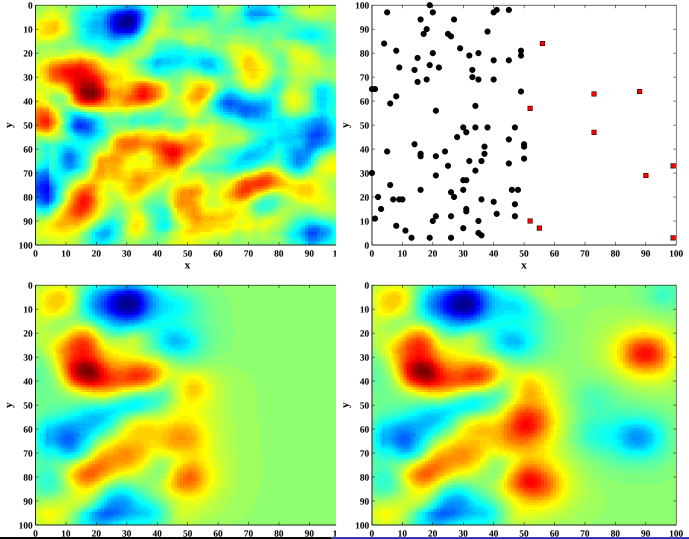


100 analog sensors and 5 digital sensors



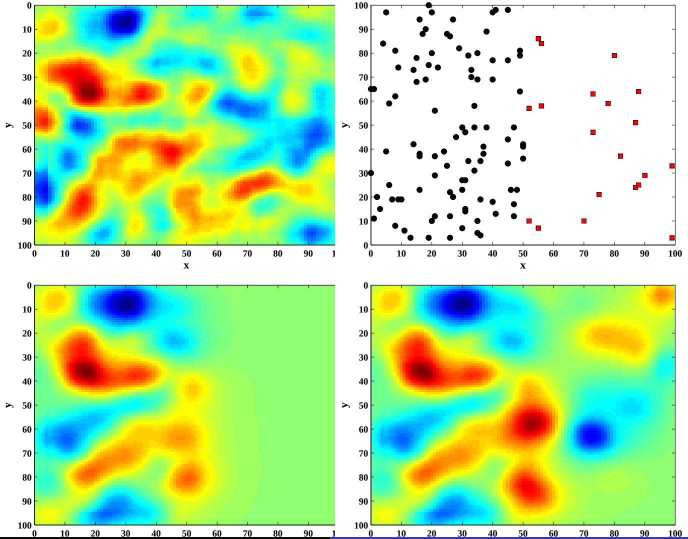
Spatial Field Reconstruction

100 analog sensors and 10 digital sensors



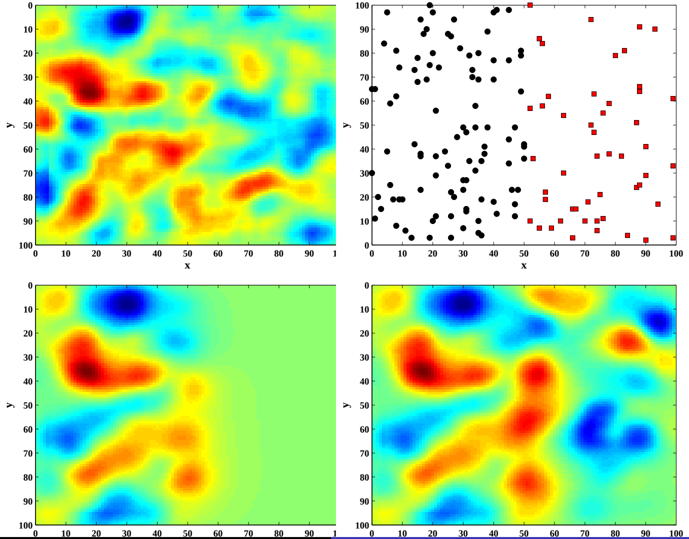
Spatial Field Reconstruction

100 analog sensors and 20 digital sensors



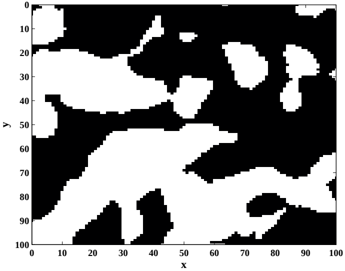
Spatial Field Reconstruction

100 analog sensors and 50 digital sensors



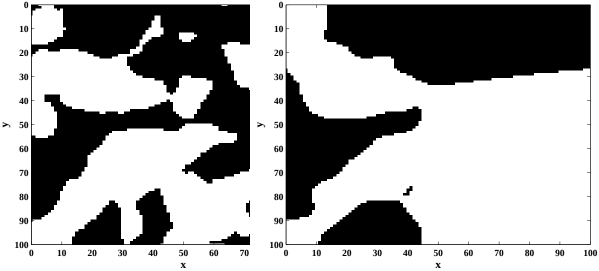
Spatial Classification

100 analog sensors and 5 digital sensors



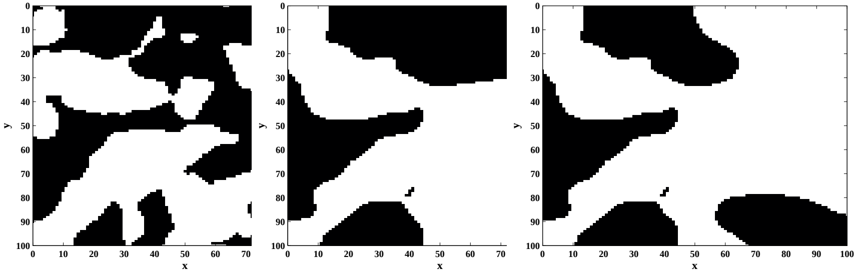
Spatial Classification

100 analog sensors and 5 digital sensors

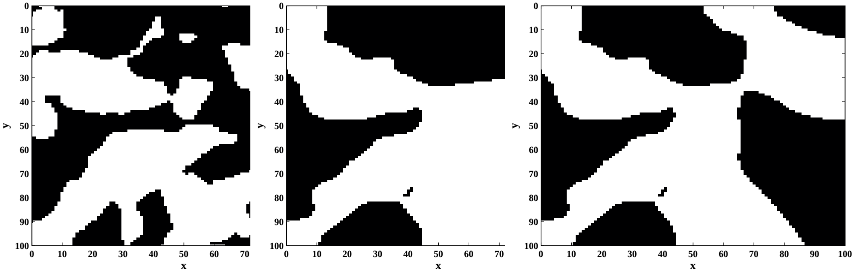


Spatial Classification

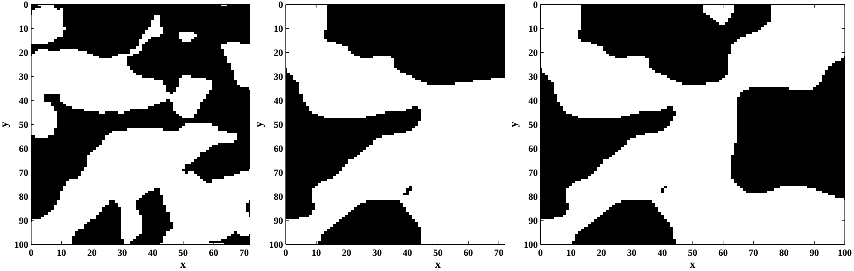
100 analog sensors and 5 digital sensors



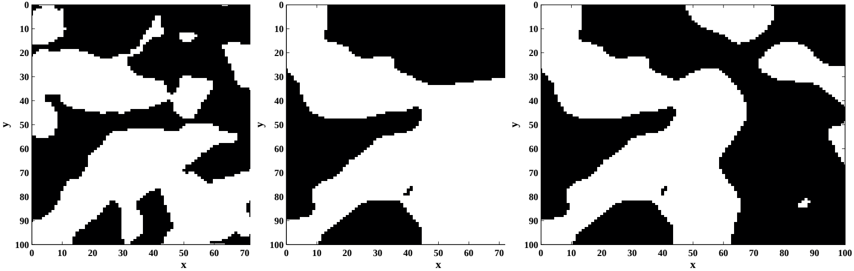
100 analog sensors and 10 digital sensors



100 analog sensors and 20 digital sensors



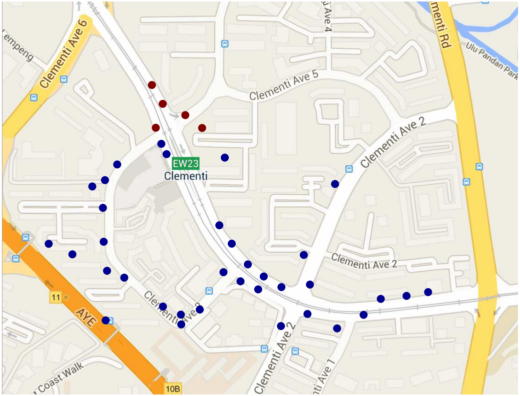
100 analog sensors and 50 digital sensors



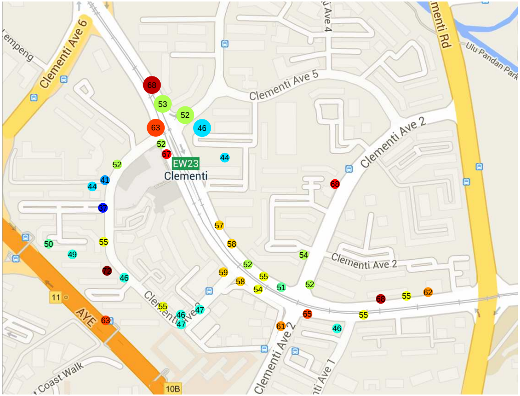
Real deployment in Singapore

Wireless sensor network deployed in Clementi to monitor acoustic intensity ("noise")

Sensors deployment

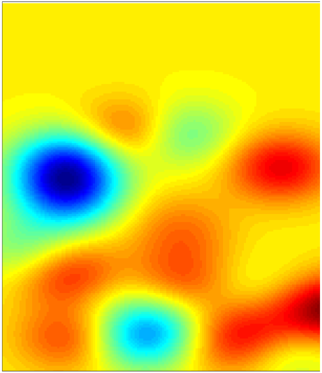


Sensors deployment

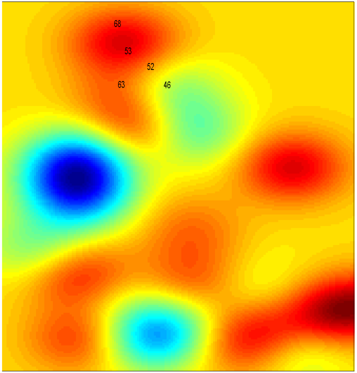


Field Reconstruction

Random field reconstruction



35 analog sensors



35 analog + 5 digital sensors

Conclusions

1. Developed a new model for sensors networks with mixed analog and digital (binary) sensors.
2. Derived the Laplace approximation to obtain the predictive posterior density.
3. Developed the spatial field reconstruction, spatial classification and spatial exceedance algorithms.
4. Simulations show the benefits of using digital sensors.

Questions?

Thanks very much!

Questions?