Networks, Random Graphs and Percolation

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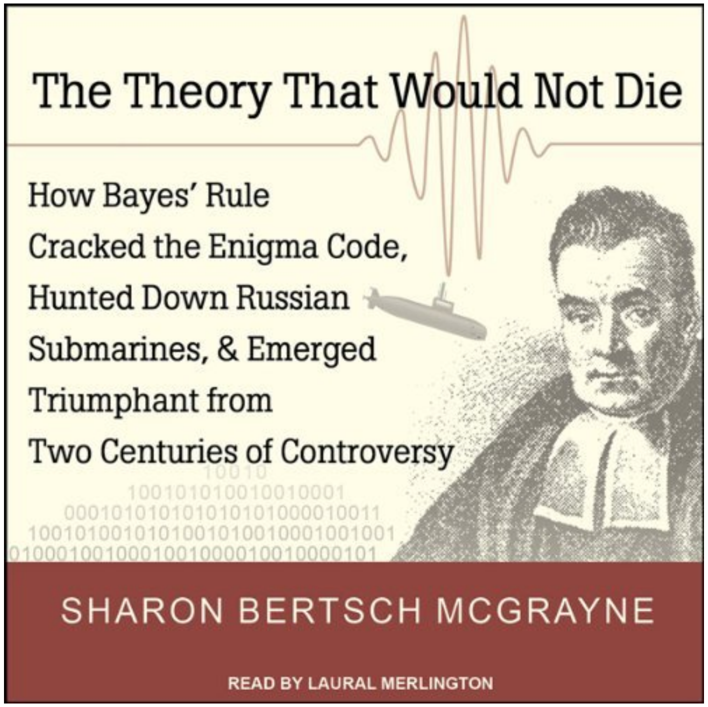
July 29, 2014

Spatial and Temporal Modeling Workshop

Institute of Statistical Mathematics ISM, Japan

Reverend Thomas Bayes (1701-1761)

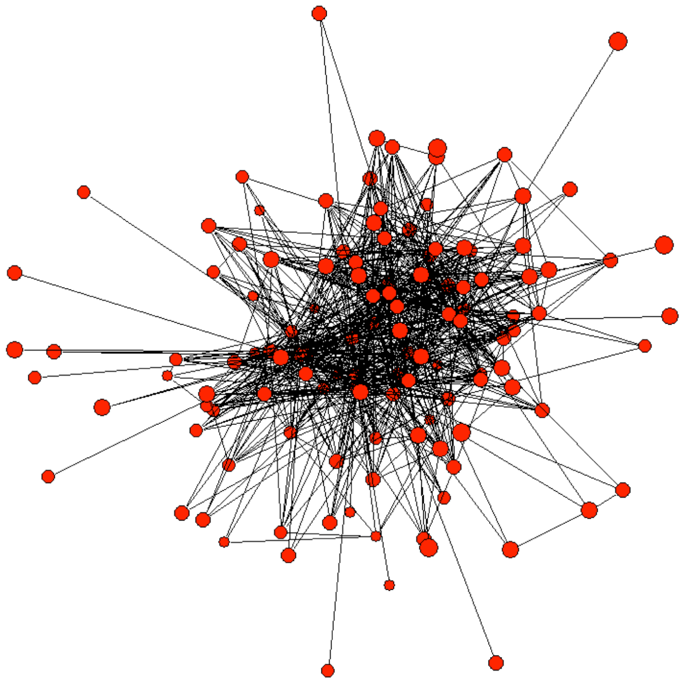


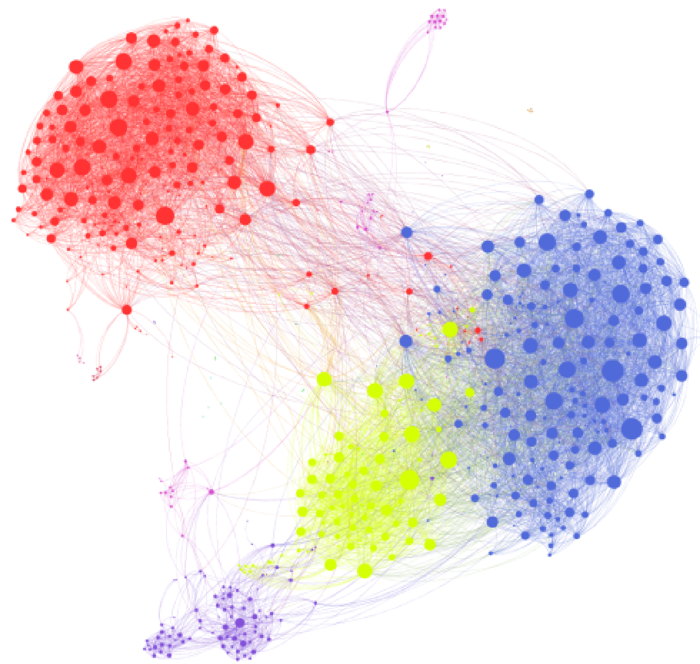
Bunhill Fields Burial Ground

London UK

Aim of this presentation

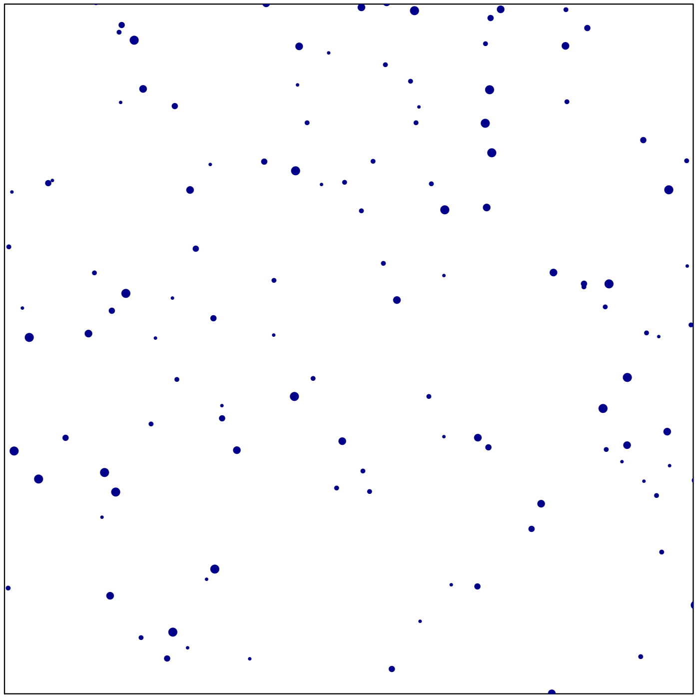
Find models that have geometric properties which are in line with the stylized facts of real world networks such as financial networks and social networks.

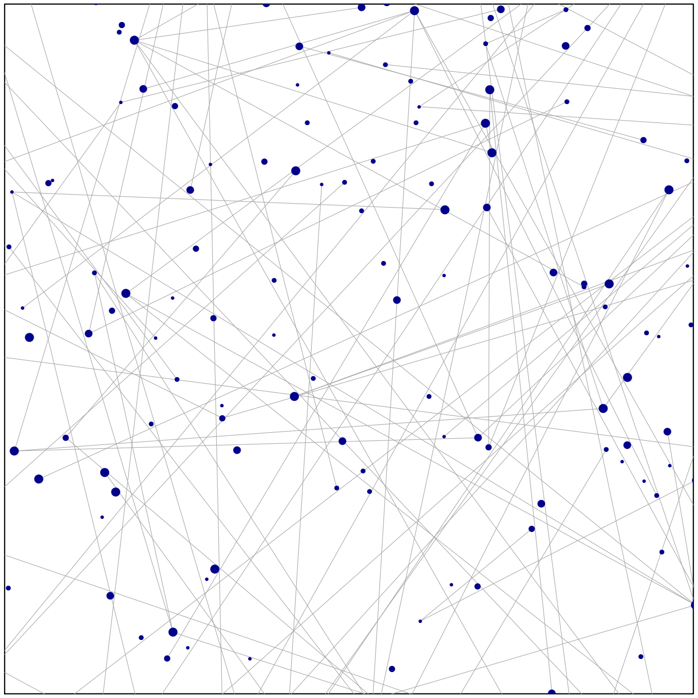




Source: (lhs) Brazilian interbank network, Cont et al. (2010); (rhs) Facebook network, griffsgraphs.com.

Graph constructions (1/2)

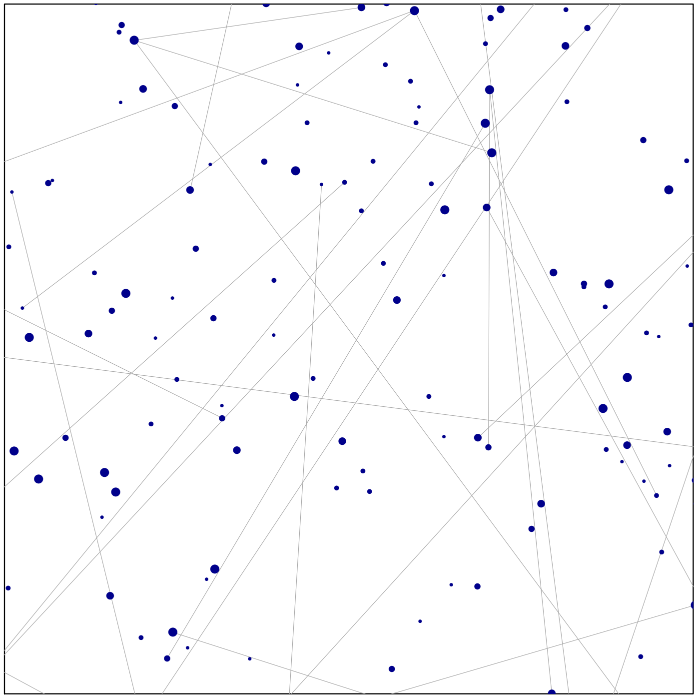


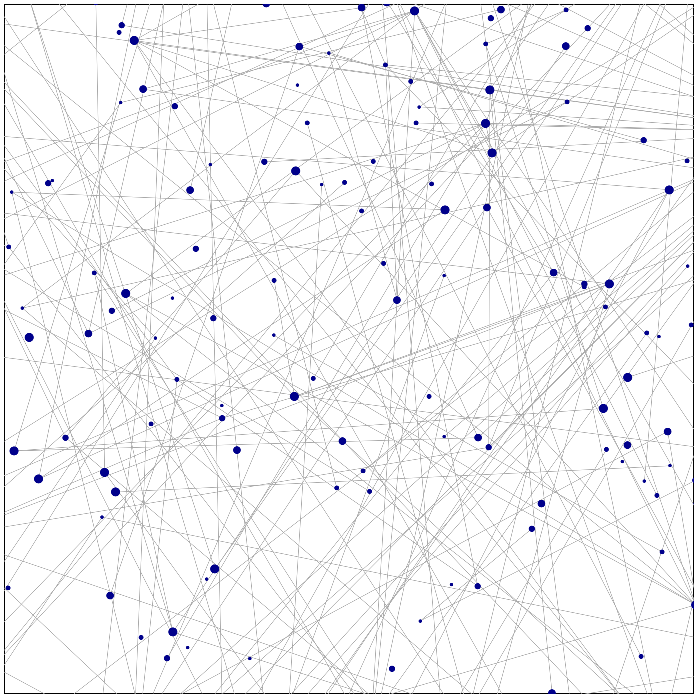


Particles $x, y \in \mathbb{R}^d$ are connected at random with given edge probabilities $p_{x,y}$.

Questions: Choice of **particles**? Choice of **edges**?

Graph constructions (2/2)

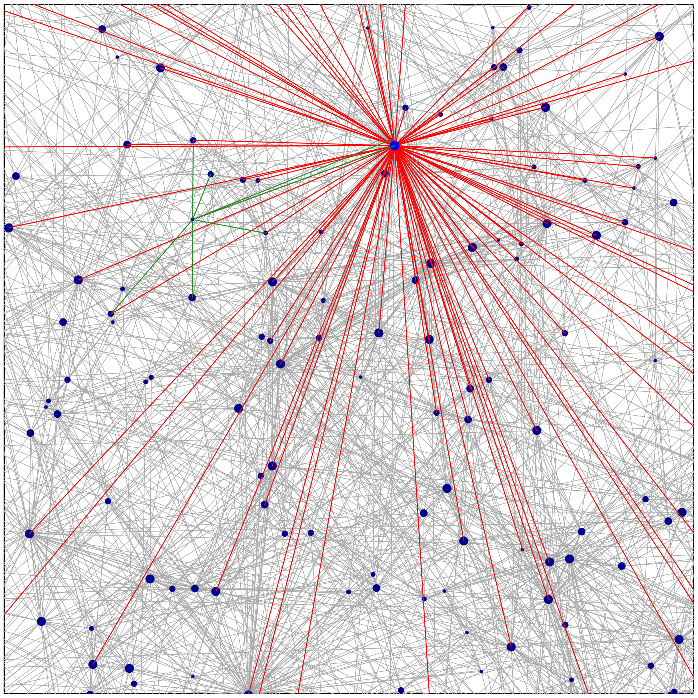




Edge probabilities $p_{x,y}$ on (lhs) are smaller than the ones on (rhs).

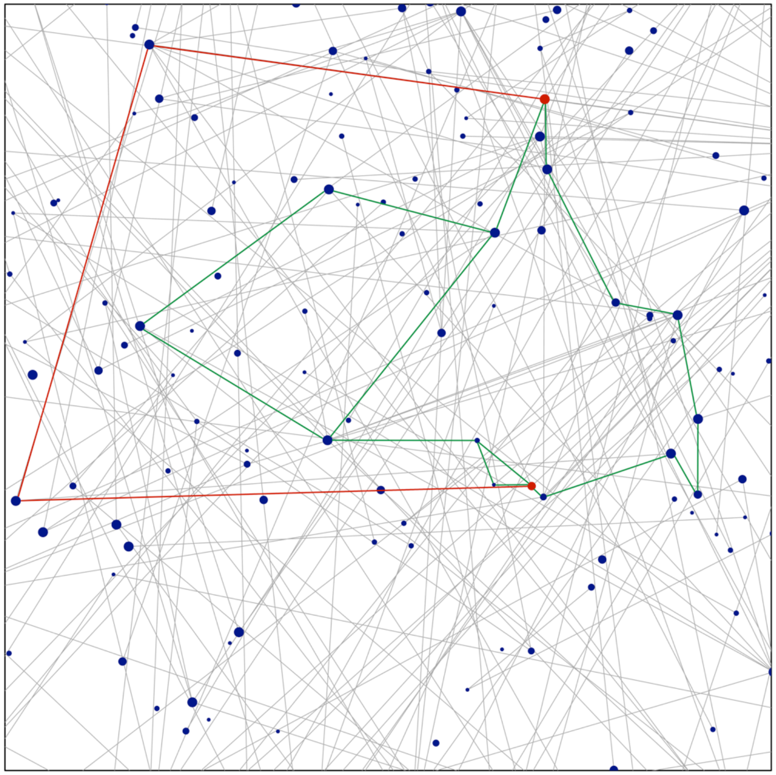
Degree distribution

Degree $\mathcal{D}(x)$ denotes the number of particles that share a direct edge with x (direct neighbors of x in the network).



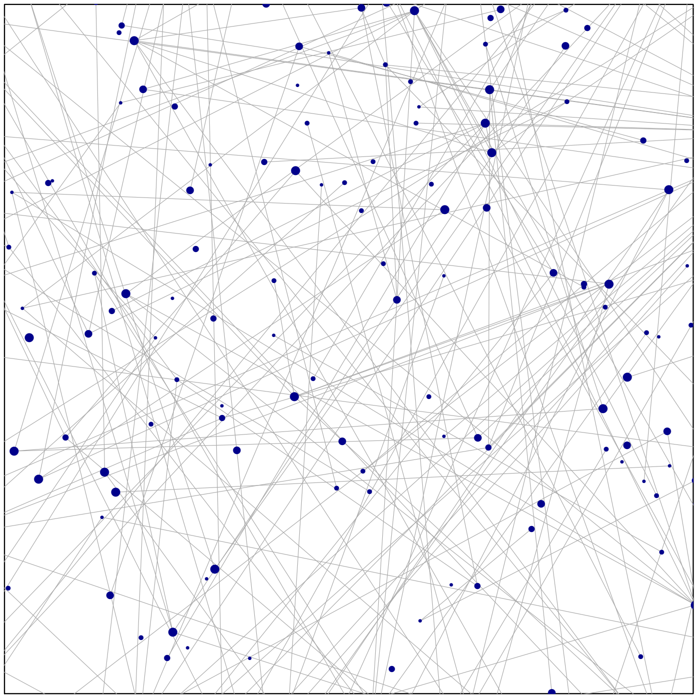
- green: particle $x_1 \in \mathbb{R}^d$ that has a low degree $\mathcal{D}(x_1)$.
- red: particle $x_2 \in \mathbb{R}^d$ that has a high degree $\mathcal{D}(x_2)$. Such particles play the role of hubs in the network.

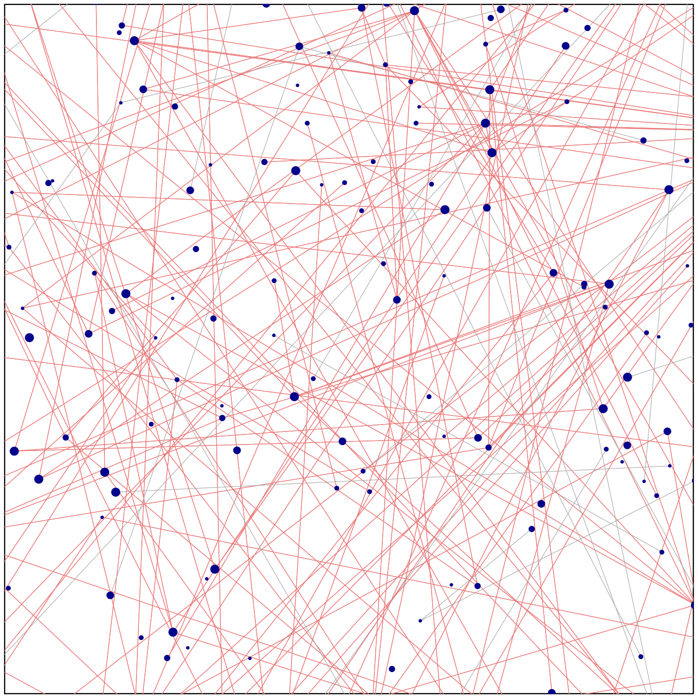
Graph distance



$d(x, y) =$ minimal number of edges that link particles x and y.

Connected components





The connected component $\mathcal{C}(x)$ is the set of particles that can be reached from x within the network, i.e.

$$\mathcal{C}(x) = \{y;\ x \text{ and } y \text{ are connected by a finite path of edges}\}.$$

$\triangleright$ $\mathcal{C}(x)$ is also called **cluster** of particle x.

Stylized facts about many real world networks

- **Small-world effect**: Any two particles are connected by very few edges. *Six Degrees* by Watts (2003) was inspired by the statement of his father saying that *"he is only 6 handshakes away from the president of the US"*.

- **Clustering property**: Connected particles tend to share common friends.

- **Power law of degrees**: The number of direct edges $\mathcal{D}(x)$ of a particle x is heavy-tailed, i.e.

$$\mathbb{P}[\mathcal{D}(x) > k] \sim ck^{-\tau} \qquad \text{as } k \to \infty,$$

with tail parameter $\tau \in (1, 2)$ (finite mean and infinite variance).

  - ⋆ number of oriented links on web pages: $\tau \approx 1.5$
  - ⋆ routers for e-mails and files: $\tau \approx 1.2$
  - ⋆ movie actor network: $\tau \approx 1.3$
  - ⋆ citation network Physical Review D: $\tau \approx 1.9$

Source: Section 1.4 in Durrett (2007) and Newman et al. (2002).

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1. Erdös-Rényi graph (1959)

2. Newman-Strogatz-Watts graph (2001)

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4. Homogeneous long-range percolation (1983)

5. Heterogeneous long-range percolation, scale-free percolation (2013)

6. Continuum space long-range percolation

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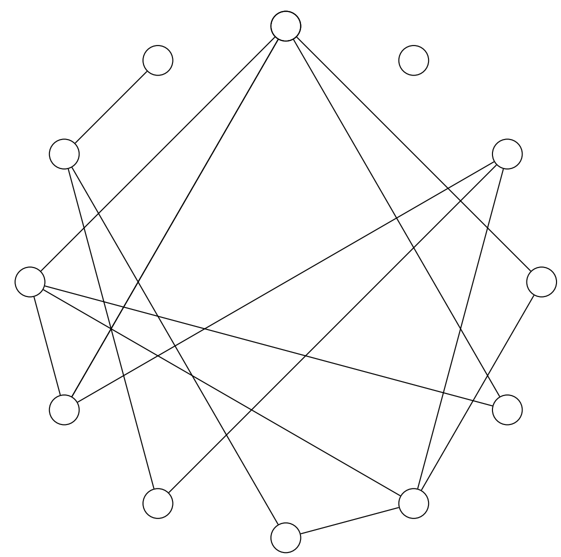
Erdös-Rényi (ER) random graph (1959)

- Choose set of n particles $V_n = \{1, \dots, n\}$.
- Fix edge probability $p \in (0, 1)$.
- Attach to $x \neq y \in V_n$ independently an edge

$$\eta_{x,y} = \eta_{y,x} = \begin{cases} 1 & \text{with } p, \\ 0 & \text{with } 1-p. \end{cases}$$

$\eta_{x,y} = 1$ means that there is an edge between x and y, i.e. x and y are adjacent.



ER graph with $n = 12$.

▷ This random graph model is usually denoted by $\mathrm{ER}(n, p)$.

▷ We consider the ER graph for large n, i.e. big sets V_n, and small $p = p_n$.

Degree distribution of ER graph

- $\mathcal{D}(x) = |\{y \in V_n;\ \eta_{x,y} = 1\}|$ degree of x.

- The degree $\mathcal{D}(x)$ fulfills for $k < n$

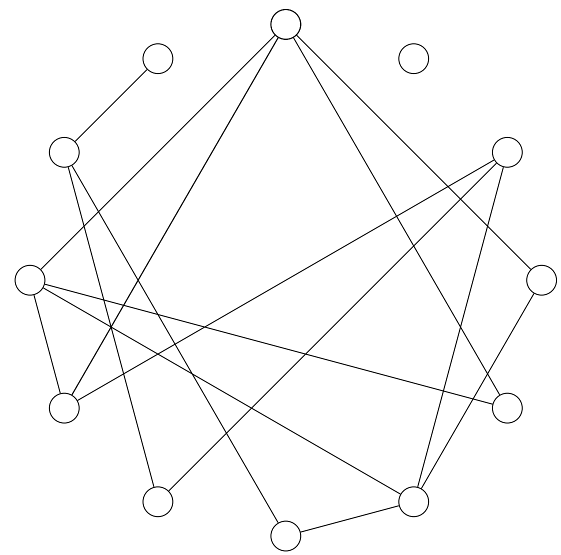
$$\mathbb{P}\left[\mathcal{D}(x) = k\right] = \binom{n-1}{k} p^k \left(1-p\right)^{n-1-k}.$$

- For $p = p_n = \vartheta/n > 0$ we obtain

$$\mathbb{P}\left[\mathcal{D}(x) = k\right] \quad \overset{n\to\infty}{\longrightarrow} \quad e^{-\vartheta}\, \frac{\vartheta^k}{k!},$$

i.e. asymptotic Poisson(ϑ) distribution.

▷ No heavy-tailed degrees $\mathcal{D}(x)$.



ER graph with $n = 12$.

Phase transition of ER graph at $\vartheta = 1$

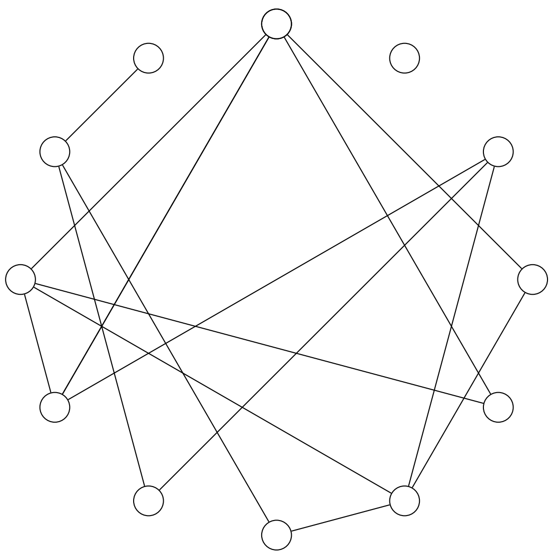
- For $p = p_n = \vartheta/n > 0$ we obtain

$$\mathbb{P}\left[\mathcal{D}(x) = k\right] \quad \stackrel{n\to\infty}{\longrightarrow} \quad e^{-\vartheta}\,\frac{\vartheta^k}{k!}.$$

- $\vartheta < 1$: connected components are of maximal order $\mathcal{O}(\log n)$, as $n \to \infty$, i.e. are small.

- $\vartheta > 1$: largest connected component is of order $\mathcal{O}(n)$, as $n \to \infty$, all others are small.

- ER graph has very few complex components.



ER graph with $n = 12$.

▷ *ER graph does not fulfill stylized facts*.

Source: Bollobás (2001) and Chapter 2 in Durrett (2007). Phase transition is closely related to branching processes.

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Newman-Strogatz-Watts (NSW) graph (2001)

- Choose set of n particles $V_n = \{1, \ldots, n\}$.

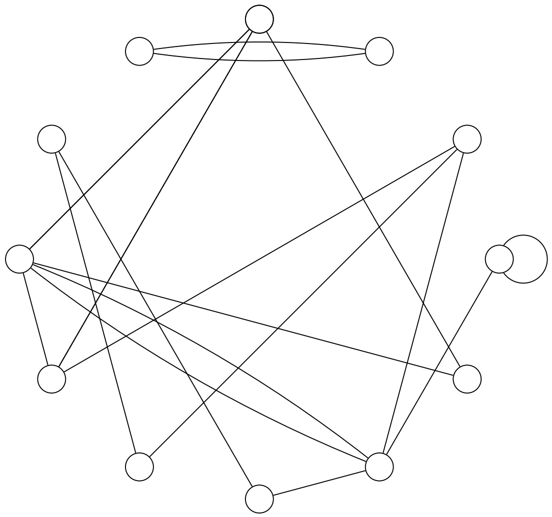
- Directly choose degree distribution for $x \in V_n$

$$g_k = \mathbb{P}\left[\mathcal{D}(x) = k\right] \sim ck^{-(\tau+1)} \text{ as } k \to \infty,$$

for fixed tail parameter $\tau > 0$.

- Note that we have the following 3 regimes:
  - ⋆ $\tau < 1$: degree $\mathcal{D}(x)$ has infinite mean;
  - ⋆ $1 < \tau < 2$: degree $\mathcal{D}(x)$ has finite mean and infinite variance;
  - ⋆ $\tau > 2$: degree $\mathcal{D}(x)$ has finite variance.

▷ These 3 regimes for τ will play a crucial role.



NSW graph with $n = 12$.

Construction of NSW graph

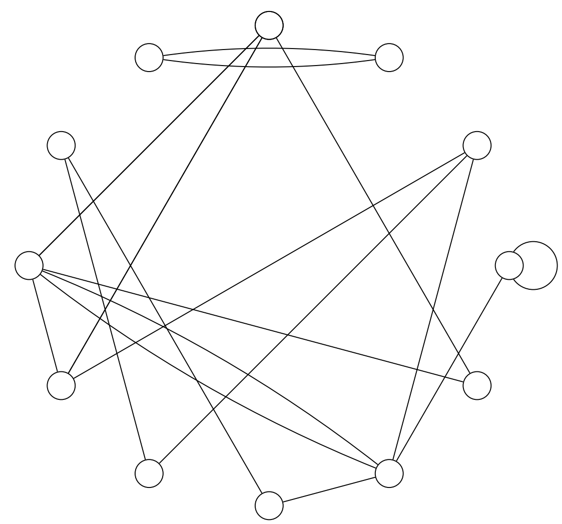
- Directly choose degrees $\mathcal{D}(x)$ according to

$$g_k = \mathbb{P}\left[\mathcal{D}(x) = k\right] \sim ck^{-(\tau+1)} \text{ as } k \to \infty.$$

- Molloy-Reed (1995) algorithm: Attach to each particle $x \in V_n$ exactly $\mathcal{D}(x)$ ends of edges and connect them randomly in pairs.

- Molloy-Reed algorithm may provide *multiple edges* and *self-loops*.

- For finite variance $\tau > 2$ there are only a few multiple edges and self-loops, as $n \to \infty$. They are described by Poisson distributions, see Theorem 3.1.2 in Durrett (2007).



NSW graph with $n = 12$.

Phase transition of NSW graph at $\vartheta = 1$

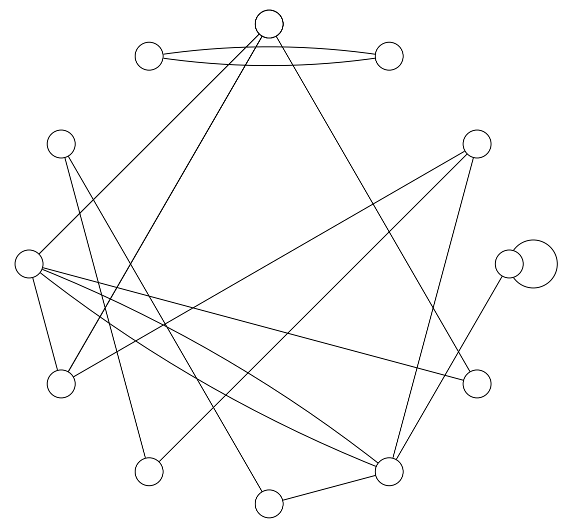
For $\tau > 1$ we define $\mu = \mathbb{E}[\mathcal{D}(x)] < \infty$ and

$$\vartheta = \mu^{-1} \sum_{k \geq 1} (k-1) k g_k.$$

- $\tau > 2$ and $\vartheta > 1$: the largest connected component is of order $\mathcal{O}(n)$, as $n \to \infty$, all others are small of order $\mathcal{O}(\log n)$.

- $\tau > 2$ and $\vartheta < 1$: connected components are conjectured to be of order $\mathcal{O}(n^{1/\tau})$, as $n \to \infty$.

- $1 < \tau < 2$: we have $\vartheta = \infty$ and the largest connected component is of order $\mathcal{O}(n)$.



NSW graph with $n = 12$.

Source: Chapter 3 in Durrett (2007).

Graph distance in NSW graphs

- Graph distance between two particles x and y

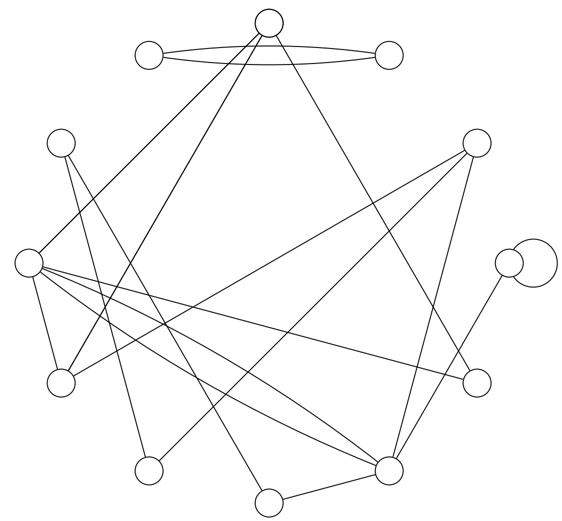
$$d(x,y) \quad = \quad \text{minimal number of edges} \\ \text{connecting } x \text{ and } y.$$

The latter is infinite if x and y are not in the same connected component.

- Van der Hofstad et al. (2007) show that $d(x,y)$ behaves for $n \to \infty$ as

$$\mathcal{O}(\log\log n) \qquad \text{for } 1 < \tau < 2,$$
$$\mathcal{O}(\log n) \qquad \text{for } \tau > 2.$$

▷ This is a small-world effect.

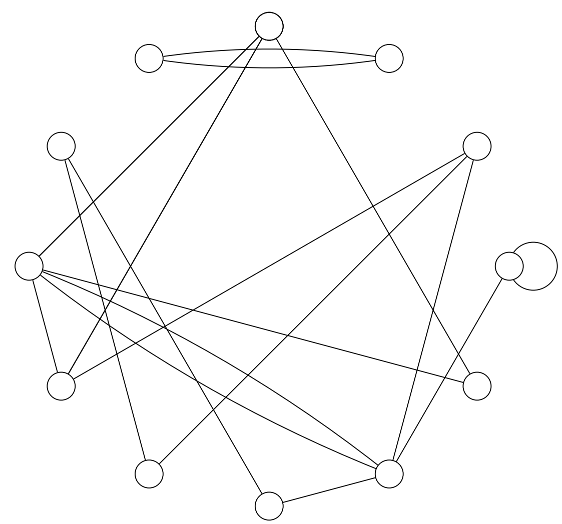


NSW graph with $n = 12$.

In fact, the statement in Van der Hofstad et al. (2007) is much more involved.

Conclusions on the NSW graph

- NSW graphs have:
  - ⋆ heavy tails, power law behavior by construction with tail parameter $\tau > 0$;
  - ⋆ small-world effect, graph distance $d(x, y)$ is of low order as $n \to \infty$;
  - ⋆ the clustering property and geometric properties are difficult to judge (for $\tau > 2$ we have "local sparsity").



NSW graph with $n = 12$.

- Aim: introduce other classes of (random) graphs that possess a natural distance function additionally to the graph topology.
- This leads to long-range percolation models in $\mathbb{Z}^d$ and $\mathbb{R}^d$.

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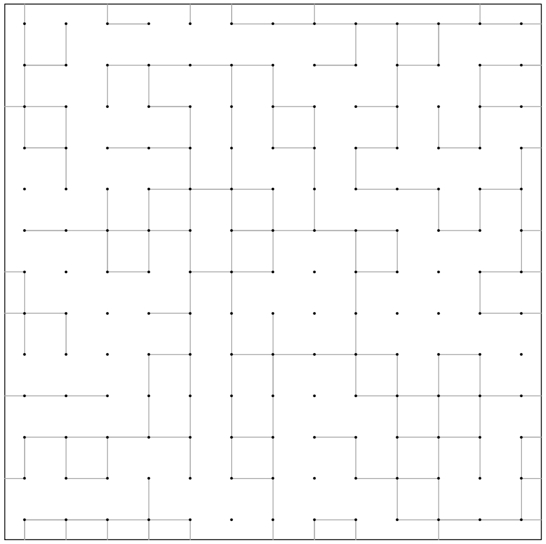
Nearest-neighbor bond percolation in $\mathbb{Z}^d$

- Denote by $\|\cdot\|$ the Euclidean distance.
- $x, y \in \mathbb{Z}^d$ nearest neighbors if $\|x - y\| = 1$.
- Fix edge probability $p \in [0, 1]$.
- Attach to $x \neq y \in \mathbb{Z}^d$ independently an edge

$$\eta_{x,y} = \eta_{y,x} = \begin{cases} 1_{\{\|x-y\|=1\}} & \text{with } p, \\ 0 & \text{with } 1-p. \end{cases}$$

$\eta_{x,y} = 1$ means that there is an edge between x and y, i.e. x and y are adjacent.



nearest-neighbor bond percolation

Source: Broadbent-Hammersley (1957), Kesten (1982), Grimmett (1997, 1999).

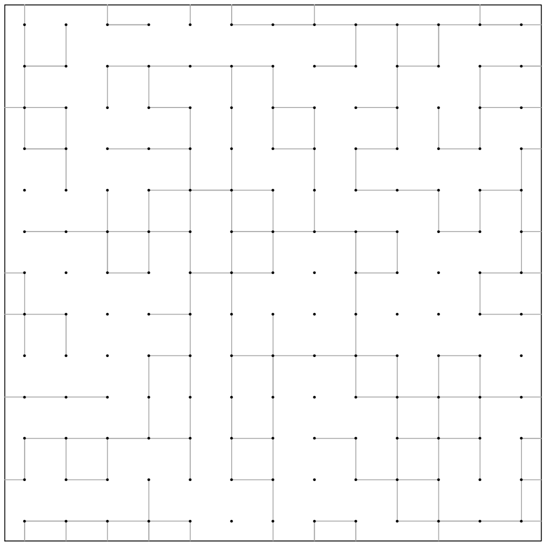
Properties: nearest-neighbor bond percolation

Properties of nearest-neighbor bond percolation:

- Degree $\mathcal{D}(x) \leq 2^d$ is bounded, and hence not heavy-tailed.

- $d(x, y) \geq \|x - y\|$, no small-world effect.

- But nearest-neighbor bond percolation in $\mathbb{Z}^d$ serves as introduction and is important for many proofs in long-range percolation.



nearest-neighbor bond percolation

- Connected component $\mathcal{C}(x)$ of $x \in \mathbb{Z}^d$:

$$\mathcal{C}(x) = \{y \in \mathbb{Z}^d;\ x \text{ and } y \text{ are connected by a finite path of nearest-neighbor edges}\}.$$

Percolation and critical probability

- Define the percolation probability

$$\theta(p) = \mathbb{P}\left[|\mathcal{C}(x)| = \infty\right].$$

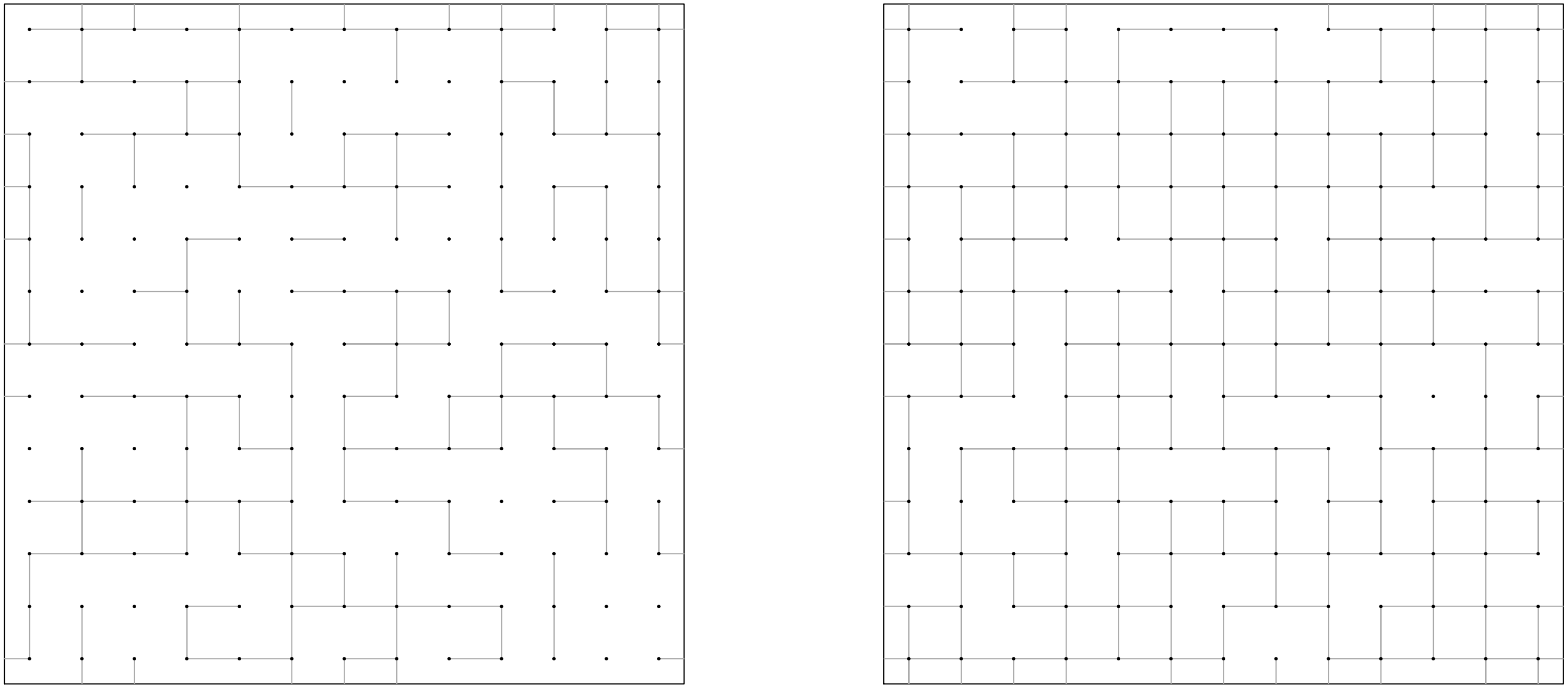
$\theta(p)$ is non-decreasing.

- Define the critical probability $p_c = p_c(\mathbb{Z}^d)$ by

$$p_c = \inf\left\{p \in (0,1];\ \theta(p) > 0\right\}.$$

- First consequences:
  - ⋆ $p_c(\mathbb{Z}^{d+1}) \leq p_c(\mathbb{Z}^d)$, because one can embed $\mathbb{Z}^d$ into $\mathbb{Z}^{d+1}$.
  - ⋆ For $p > p_c$ we have $\theta(p) > 0$ and $x \in \mathbb{Z}^d$ belongs to an infinite connected component $\mathcal{C}(x)$ with positive probability.

Critical probability for $d = 2$



Nearest-neighbor bond percolation in dimension $d = 2$: $p_c = p_c(\mathbb{Z}^2) = 1/2$.

(lhs) $p < p_c$; (rhs) $p > p_c$.

Source: Duality argument of Kesten.

Phase transition picture

Critical probability

$$p_c = p_c(\mathbb{Z}^d) = \inf \{p \in (0,1];\ \theta(p) > 0\}\,.$$

Theorem 1. For nearest-neighbor bond percolation in $\mathbb{Z}^d$ we have

- $d = 1$: $p_c(\mathbb{Z}) = 1$; and
- $d \geq 2$: $p_c(\mathbb{Z}^d) \in (0,1)$.

Denote by $\mathcal{I}$ the number of infinite connected components.

Theorem 2. For any $p \in (0,1)$ either $\mathbb{P}[\mathcal{I} = 0] = 1$ or $\mathbb{P}[\mathcal{I} = 1] = 1$.

$\Longrightarrow$ **For $p > p_c(\mathbb{Z}^d)$ there exists a *unique* infinite connected component $\mathcal{C}_\infty$, a.s.**

Conclusion. Nearest-neighbor bond percolation does not share the stylized facts.

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Homogeneous long-range percolation in $\mathbb{Z}^d$

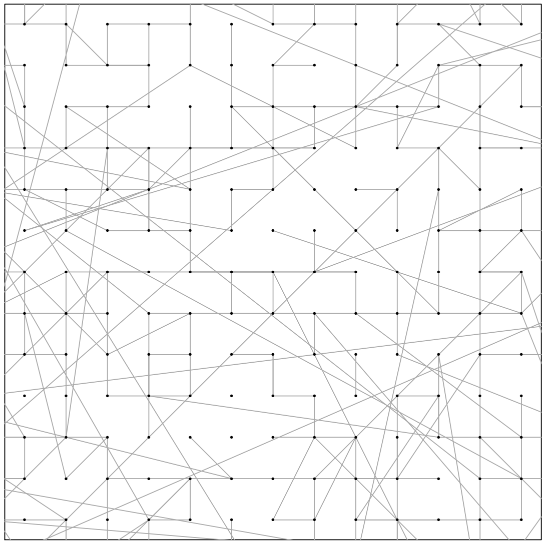
- Fix edge probabilities

$$p_{x,y} = \begin{cases} p & \text{if } \|x-y\| = 1, \\ 1 - e^{-\lambda \|x-y\|^{-\alpha}} & \text{if } \|x-y\| > 1, \end{cases}$$

for given $p \in [0,1]$, $\alpha > 0$ and $\lambda > 0$.

- Attach to $x \neq y \in \mathbb{Z}^d$ independently an edge

$$\eta_{x,y} = \eta_{y,x} = \begin{cases} 1 & \text{with } p_{x,y}, \\ 0 & \text{with } 1 - p_{x,y}. \end{cases}$$



homogeneous long-range percolation

▷ $p_{x,y} \sim \lambda \|x-y\|^{-\alpha}$ as $\|x-y\| \to \infty$.

Source: Schulman (1983), Newman-Schulman (1986), Gandolfi et al. (1992), Berger (2002, 2008), Benjamini et al. (2004), Biskup (2004), Trapman (2010).

Degrees for long-range percolation model

$$\mathcal{D}(x) = |\{y \in \mathbb{Z}^d : \ \eta_{x,y} = 1\}|.$$

Theorem 3. For homogeneous long-range percolation on $\mathbb{Z}^d$ we have

- $\alpha \leq d$: $\mathcal{D}(x) = \infty$ and the infinite connected component $\mathcal{C}_\infty$ contains all particles $z \in \mathbb{Z}^d$, a.s.;
- $\alpha > d$: $\mathcal{D}(x)$ behaves as a Poisson distribution, in particular, is light-tailed.

The second statement follows from thinning the lattice $\mathbb{Z}^d$ by non-adjacent particles leading to considerations of inhomogeneous Poisson point processes in $\mathbb{R}^d$.

Percolation picture and phase transitions

$$\mathcal{C}(x) = \{y \in \mathbb{Z}^d;\ x \text{ and } y \text{ are connected by a finite path of edges}\}.$$

Theorem 4. For homogeneous long-range percolation on $\mathbb{Z}^d$ we have, a.s.,

- $\alpha \leq d$: there is an infinite connected component;

- $\alpha > d$ and $d \geq 2$: for p sufficiently close to 1 there is an infinite connected component;

- $\alpha > d$ and $d = 1$:

  ⋆ $1 < \alpha < 2$: for p sufficiently close to 1 there is an infinite connected component;
  ⋆ $\alpha > 2$: there is no infinite connected component.

The case $d = 1$ and $\alpha = 2$ is also solved and percolation depends on the choice of $\lambda > 0$. Recall:

$$p_{x,y} \;=\; p \cdot 1_{\{\|x-y\|=1\}} \;+\; \left(1 - e^{-\lambda\|x-y\|^{-\alpha}}\right) \cdot 1_{\{\|x-y\|>1\}}.$$

Graph distances in long-range percolation

$d(x, y) =$ minimal number of edges that connect x and y.

Theorem 5. For homogeneous long-range percolation on $\mathbb{Z}^d$ we have

- $\alpha < d$: the graph distance is bounded, a.s., by

$$\lceil d/(d-\alpha) \rceil \, ;$$

- $d < \alpha < 2d$: assume, a.s., that there exists a unique infinite connected component $\mathcal{C}_\infty$.

For all $\epsilon > 0$ we have, set $\Delta^{-1} = \log_2(2d/\alpha)$,

$$\lim_{\|x\| \to \infty} \mathbb{P}\left[\Delta - \epsilon \leq \frac{\log d(0,x)}{\log \log \|x\|} \leq \Delta + \epsilon \,\middle|\, 0, x \in \mathcal{C}_\infty \right] = 1;$$

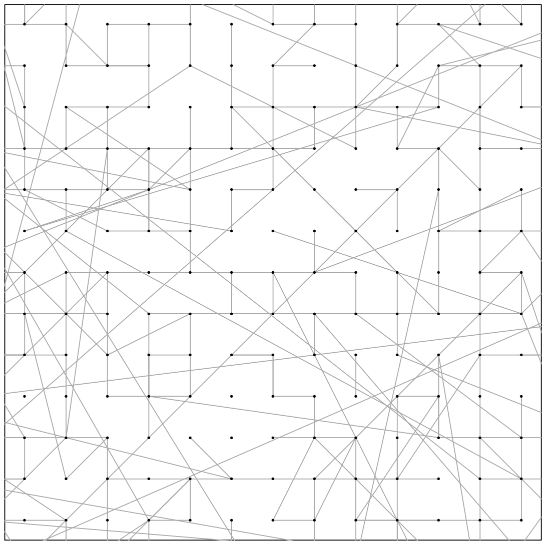
- $\alpha > 2d$: we have, a.s.,

$$\liminf_{\|x\| \to \infty} \frac{d(0,x)}{\|x\|} > 0.$$

Homogeneous long-range percolation in $\mathbb{Z}^d$

For $p_{x,y} \sim \lambda \|x-y\|^{-\alpha}$ as $\|x-y\| \to \infty$:

- $\alpha \leq d$: $\mathcal{C}_\infty$ contains all particles of $\mathbb{Z}^d$, degrees are infinite, graph distances are bounded, a.s.;
- $d < \alpha < 2d$: degrees $\mathcal{D}(x)$ are light-tailed, local clustering, small-world effect;
- $\alpha > 2d$: behaves as nearest-neighbor bond percolation on $\mathbb{Z}^d$.



homogeneous long-range percolation

Conclusion. The homogeneous long-range percolation model in $\mathbb{Z}^d$ shares many good properties for $d < \alpha < 2d$, except of the heavy-tailedness of the degree distribution.

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Heterogeneous long-range percolation in $\mathbb{Z}^d$

- $(W_x)_{x\in\mathbb{Z}^d}$ are i.i.d. Pareto$(1,\beta)$ with $\beta > 0$,

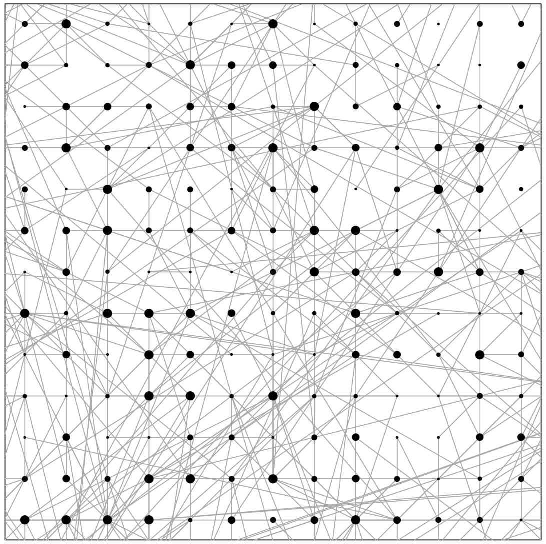
$$\mathbb{P}[W_x > w] = w^{-\beta}, \quad \text{for } w \geq 1.$$

- For fixed $\alpha, \lambda > 0$ and given $(W_x)_{x\in\mathbb{Z}^d}$ set

$$p_{x,y} = 1 - e^{-\lambda W_x W_y \|x-y\|^{-\alpha}}.$$

- Choose independently edges for $x \neq y \in \mathbb{Z}^d$

$$\eta_{x,y} = \eta_{y,x} = \begin{cases} 1 & \text{with } p_{x,y}, \\ 0 & \text{with } 1 - p_{x,y}. \end{cases}$$



heterogeneous long-range percolation

$\triangleright$ $p_{x,y} \approx \lambda W_x W_y \|x-y\|^{-\alpha}$.

Source: Deijfen et al. (2013).

Degrees for heterogeneous long-range percolation

Theorem 6. For heterogeneous long-range percolation on $\mathbb{Z}^d$ we have

- $\min\{\alpha, \beta\alpha\} \leq d$: $\mathcal{D}(x) = \infty$, a.s.;
- $\min\{\alpha, \beta\alpha\} > d$: set $\tau = \beta\alpha/d > 1$. Then

$$\mathbb{P}\left[\mathcal{D}(x) > k\right] = k^{-\tau}\ell(k) \qquad \text{as } k \to \infty,$$

for some function $\ell(\cdot)$ that is slowly varying at infinity.

$\Longrightarrow$ The second statement provides heavy-tailedness of degrees! Compare to Theorem 3.

- $\tau < 2$: infinite variance of degrees $\mathcal{D}(x)$.
- $\tau > 2$: finite variance of degrees $\mathcal{D}(x)$.

Percolation picture and phase transitions

Fix $\alpha, \beta > 0$. Define critical constant

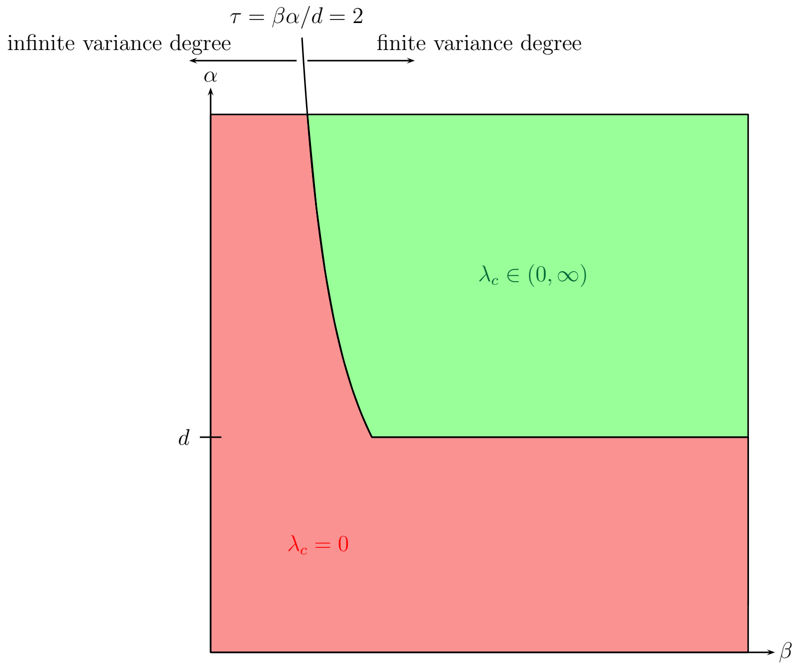
$$\lambda_c = \inf \{\lambda > 0;\ \mathbb{P}[|\mathcal{C}(x)| = \infty] > 0\}.$$

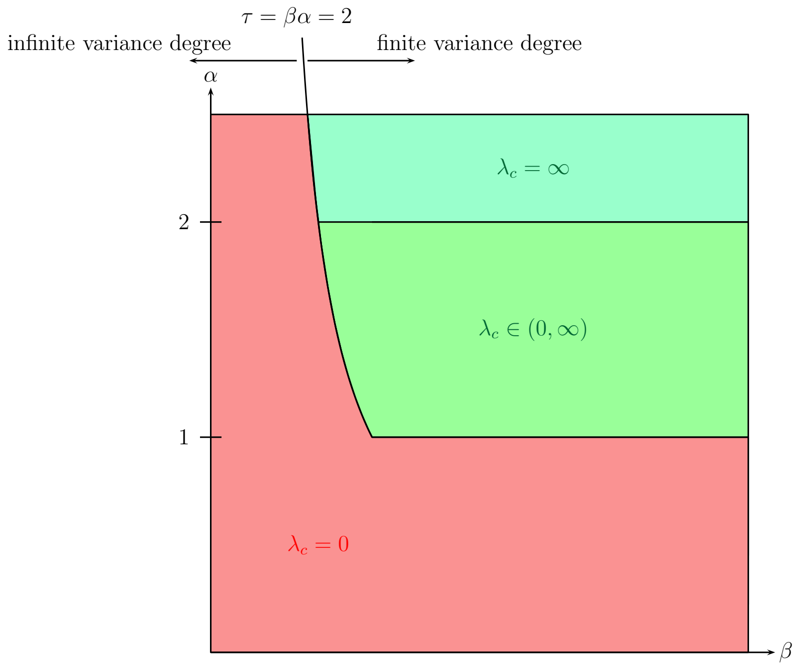
Theorem 7. Fix $d \geq 1$ and assume $\min\{\alpha, \beta\alpha\} > d$. This implies $\tau > 1$.

- Upper bounds:
  - ⋆ $d \geq 2$: $\lambda_c < \infty$;
  - ⋆ $d = 1$ and $\alpha \in (1, 2]$: $\lambda_c < \infty$;
  - ⋆ $d = 1$ and $\min\{\alpha, \beta\alpha\} > 2$: $\lambda_c = \infty$.
- Lower bounds:
  - ⋆ $\tau = \beta\alpha/d < 2$ (infinite variance): $\lambda_c = 0$;
  - ⋆ $\tau = \beta\alpha/d > 2$ (finite variance): $\lambda_c > 0$.

This is similar to Theorem 4 of homogeneous long-range percolation.

Phase transition picture





(lhs) phase transition for $d \geq 2$; (rhs) phase transition for $d = 1$.

Graph distances in heterogeneous model (1/2)

Theorem 8. We have for $\min\{\alpha, \beta\alpha\} > d$:

- $1 < \tau < 2$ (infinite variance): for $\lambda > \lambda_c = 0$

$$\lim_{\|x\|\to\infty} \mathbb{P}\left[\eta_1 \leq \frac{d(0,x)}{\log\log\|x\|} \leq \eta_2 \middle| 0, x \in \mathcal{C}_\infty\right] = 1;$$

- $d < \alpha < 2d$ and $\tau > 2$ (finite variance): for $\lambda > \lambda_c \in (0, \infty)$

$$\lim_{\|x\|\to\infty} \mathbb{P}\left[\eta_3 \leq \frac{\log d(0,x)}{\log\log\|x\|} \leq \eta_4 \middle| 0, x \in \mathcal{C}_\infty\right] = 1;$$

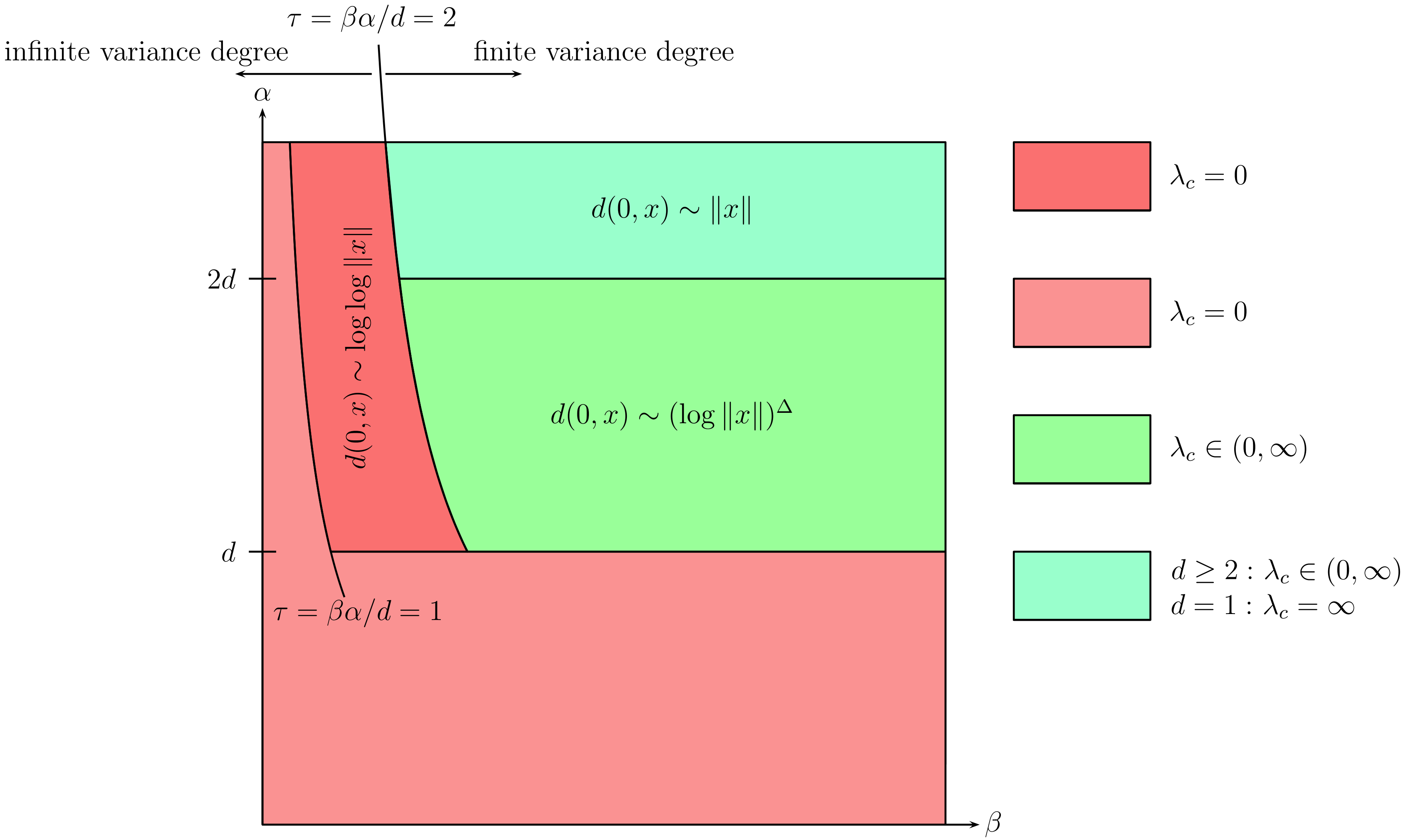
- $\alpha > 2d$ and $\tau > 2$ (finite variance):

$$\lim_{\|x\|\to\infty} \mathbb{P}\left[\eta_5 \leq \frac{d(0,x)}{\|x\|}\right] = 1.$$

- Statements $\leq$ are not rigorously proved.

- Compare to Theorem 5: Case $1 < \tau < 2$ is new!

Graph distances in heterogeneous model (2/2)

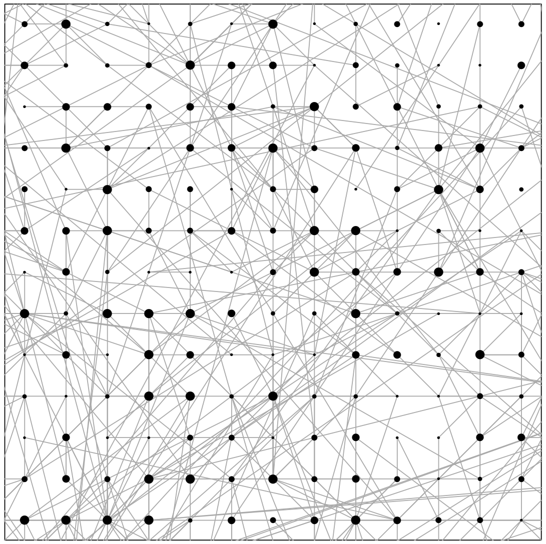


Case $1 < \tau = \beta\alpha/d < 2$ (infinite variance of degrees) is new compared to homogeneous long-range percolation. This provides small-world effect of order $\log\log \|x\|$.

Heterogeneous long-range percolation in $\mathbb{Z}^d$

For $p_{x,y} \approx \lambda W_x W_y \|x - y\|^{-\alpha}$:

- $\min\{\alpha, \beta\alpha\} \leq d$: degree $\mathcal{D}(x)$ is infinite, a.s.
- $\min\{\alpha, \beta\alpha\} > d$: degree $\mathcal{D}(x)$ has power law with parameter $\tau = \beta\alpha/d > 1$.
- $d < \min\{\alpha, \beta\alpha\} < 2d$: small-world effect and local clustering.
- $\min\{\alpha, \beta\alpha\} > 2d$: conjectured to be as nearest-neighbor bond percolation.



heterogeneous long-range percolation

Conclusions. The heterogeneous long-range percolation model in $\mathbb{Z}^d$ shares the stylized facts for $d < \min\{\alpha, \beta\alpha\} < 2d$, in particular, $1 < \tau = \beta\alpha/d < 2$ is attractive for real world modeling.

Contents

1. Erdös-Rényi graph (1959)

2. Newman-Strogatz-Watts graph (2001)

3. Nearest-neighbor Bernoulli bond percolation (1957)

4. Homogeneous long-range percolation (1983)

5. Heterogeneous long-range percolation, scale-free percolation (2013)

6. Continuum space long-range percolation

Continuum space long-range percolation in $\mathbb{R}^d$

- Particles $X \subset \mathbb{R}^d$ come from a homogeneous Poisson cloud in $\mathbb{R}^d$.

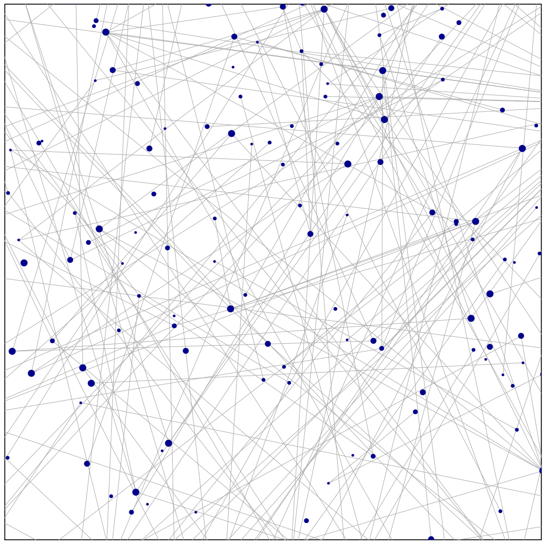
- $(W_x)_{x \in X}$ are i.i.d. Pareto$(1, \beta)$ marks of X.

- For fixed $\alpha, \lambda > 0$ and given X and $(W_x)_{x \in X}$:

$$p_{x,y} = 1 - e^{-\lambda W_x W_y \|x-y\|^{-\alpha}}.$$

- Choose independently edges for $x \neq y \in X$

$$\eta_{x,y} = \eta_{y,x} = \begin{cases} 1 & \text{with } p_{x,y}, \\ 0 & \text{with } 1 - p_{x,y}. \end{cases}$$

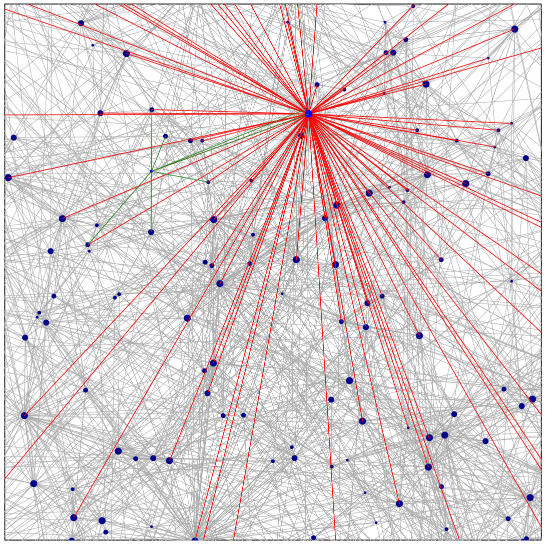


continuum space model

Continuum space long-range percolation in $\mathbb{R}^d$

Statements and conjectures:

- $\min\{\alpha, \beta\alpha\} \leq d$: degree $\mathcal{D}(x)$ is infinite, a.s.
- $\min\{\alpha, \beta\alpha\} > d$: degree $\mathcal{D}(x)$ has power law with parameter $\tau = \beta\alpha/d > 1$.
- $d < \min\{\alpha, \beta\alpha\} < 2d$: small-world effect and local clustering.
- $\min\{\alpha, \beta\alpha\} > 2d$: conjectured to be as nearest-neighbor bond percolation.



continuum space model

Conjecture. The continuum space long-range percolation model in $\mathbb{R}^d$ shares the stylized facts for $d < \min\{\alpha, \beta\alpha\} < 2d$, in particular, $1 < \tau = \beta\alpha/d < 2$ is attractive for real world modeling.

Statements about degrees $\mathcal{D}(x)$ are proved in Deprez-W. (2013).

Contents

1. Erdös-Rényi graph (1959)

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6. Continuum space long-range percolation

7. Proofs: renormalization techniques

Proofs are based on renormalization (1/3)

Define generations of boxes:

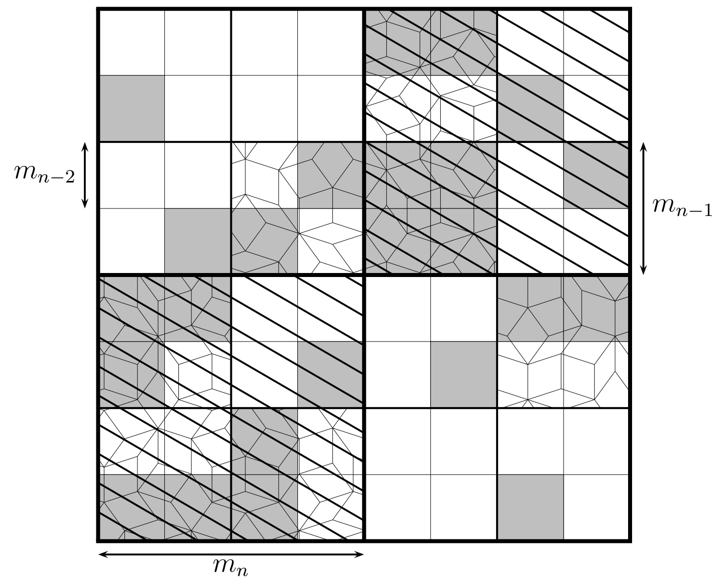
- Choose an integer valued sequence $(a_n)_{n\in\mathbb{N}_0}$ with $a_n > 1$.

- Define box lengths $(m_n)_{n\in\mathbb{N}_0}$ by

$$m_n = a_n m_{n-1} = \prod_{i=0}^{n} a_i.$$

- Choose $v \in \mathbb{Z}^d$. Box $B_{n,v}$ of generation n is defined by

$$B_{n,v} = m_n v + [0, m_n - 1]^d.$$



box generation n for $a_n \equiv 2$

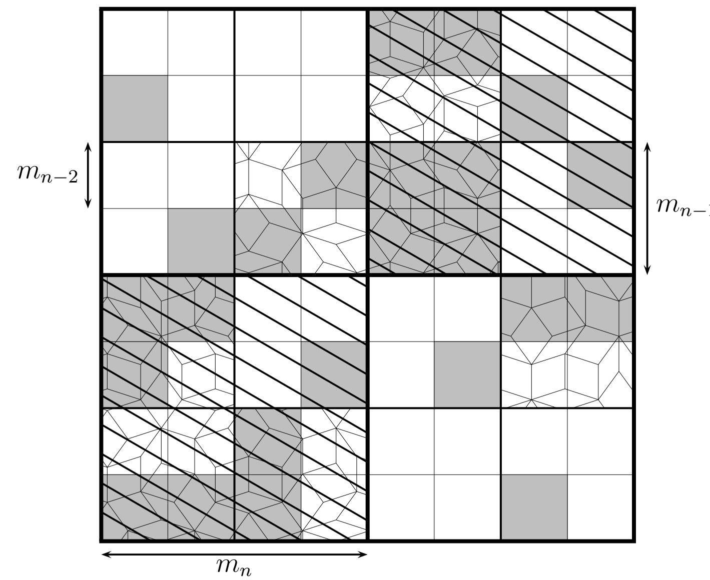
▷ Every box $B_{n,v}$ of generation n contains a_n^d children $B_{n-1,w}$ of generation $n-1$.

Proofs are based on renormalization (2/3)

Recursive algorithm for good boxes:

Choose densities $(\kappa_n)_{n\in\mathbb{N}_0}$ with $\kappa_n \in (0,1)$.

- Generation 0 box is good if it contains a connected component of size $\kappa_0 a_0^d$.
- Generation n box is good if
  - ⋆ it contains at least $\kappa_n a_n^d$ good generation $n-1$ boxes; and
  - ⋆ all good generation $n-1$ boxes are attached by a direct edge.



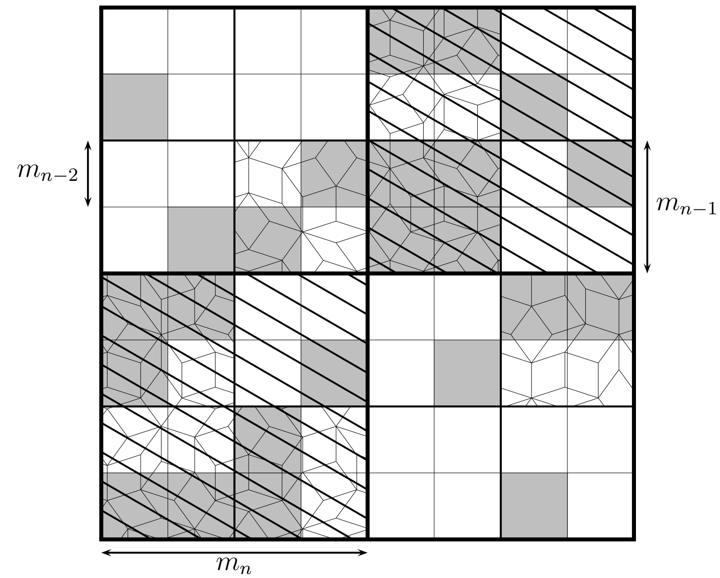
good generation n boxes $B_{n,v}$

▷ The previous algorithm builds up recursively good boxes $B_{n,v}$ of generation n which are linked through generation $n+1$ of good boxes $B_{n+1,w}$.

Proofs are based on renormalization (3/3)

Recursive algorithm for good boxes:

- Generation 0 box is good if it contains a connected component of size $\kappa_0 a_0^d$.
- Generation n box is good if
  - ⋆ it contains at least $\kappa_n a_n^d$ good generation $n-1$ boxes; and
  - ⋆ all good generation $n-1$ boxes are attached by a direct edge.



good generation n boxes $B_{n,v}$

Assume we arrive at a generation n of boxes $B_{n,v}$ such that

$$(1) \quad \mathbb{P}[\text{box } B_{n,v} \text{ is good}] > p^*,$$

$$(2) \quad \mathbb{P}[\text{boxes } B_{n,v} \text{ and } B_{n,w} \text{ are attached}] > 1 - e^{-\lambda^* \|v-w\|^{-\alpha}},$$

where we have site-bond percolation for p^* and λ^*. Then, we obtain an infinite cluster of good boxes and hence the original model also percolates (through the attachedness). □

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