Bimeausres, spectral measures and other characerization of heavy tail processes

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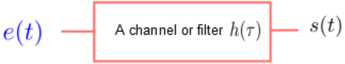
workshop on Complex systems Modeling and Estimation Challenges in big data (CSM 2014)
The Institute of Statistical mathematics (ISM)

Laboratoire de MATHEMATIQUES
UMR 6620 UBP-CNRS

Motivation and application examples

Consider a channel or a filter,



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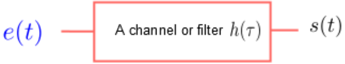

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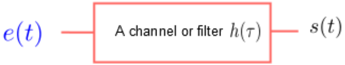

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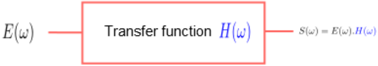

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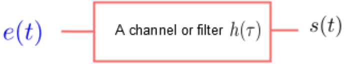

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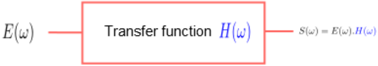

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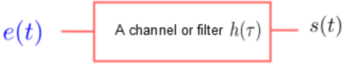

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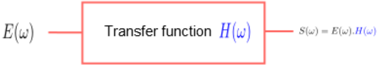

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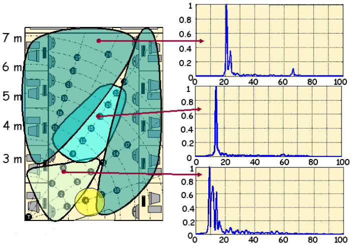

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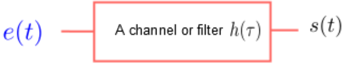

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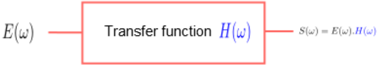

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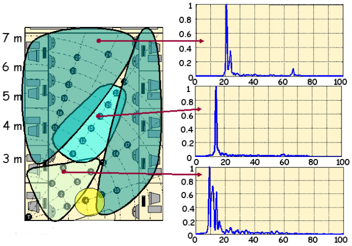

$$h(t) = \sum_{k=1}^{N} a_k \delta_{t-\tau_k} e^{i\theta_k},$$

But real world is random and $(h(t), t \geq 0)$ is considered as a stochastic process

$\Longrightarrow$ A harmonizable process

$$H(\omega) = \int e^{\iota\omega t}d\xi(t)$$

(ξ_t) (resp $d\xi(.)$) is heavy tailed process (resp. random measure)

Why α-stables?

Theoretical interest

- It is an extension of gaussian distributions and processes (case $\alpha = 2$)
- The convolution stability: a combination of i.i.d stable variables is a stable one
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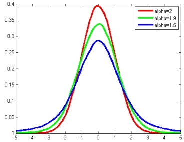
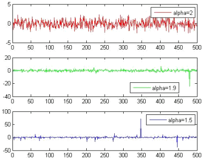
Practical modelings





- Heavier tail with the decrease of α.
- α-stables take into account extreme values usually seen as outliers for Gaussians.
- α-stable are better models the high variability phenomena (infinite variance, impulsive signals...).

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α-stable variables

1. A random variable X is said stable (or have a stable distribution) if and only if for any positive real A and B their exist a unique positive C and real D s.t:

$$AX_1 + BX_2 =^d CX + D$$

   X_1 and X_2 i.i.d copies of X (in the symmetric case D=0)

2. It was shown that in this cas there exist a unique $0 \leq \alpha \leq 2$ such that C is given by

$$A^\alpha + B^\alpha = C^\alpha$$

   Hence the prefix α

3. the characteristic function of Symmetric α-stable variables (SαS) is given by:

$$\phi_x(\theta) = \boldsymbol{E}[e^{\imath\theta X}] = e^{-\sigma^\alpha |\theta|^\alpha} \text{ where } 0 < \alpha \leq 2 \text{ and } \sigma > 0.$$

4. Unfortunately the form of the characteristic function suggest that the density function of these distribution is impossible to calculate except for three special cases ($\alpha = 1, 2,$ or $\frac{1}{2}$)

α-stable random vectors

1. A random vector $X = (X_1, \ldots, X_d)$ is α-stable SαS if for every A and B positives, their exist $C > 0$ such that:
$$AX^{(1)} + BX^{(2)} \stackrel{d}{=} CX,$$
where $X^{(1)}$ and $X^{(2)}$ are i.i.d. copies of X and $A^\alpha + B^\alpha = C^\alpha$
2. equivalently we can show that the vector X is symmetric α-stable if and only if every linear combinaison $Y = \sum_{k=1}^{d} b_k X_k$ is a an $S\alpha S$ univariate variable.
3. The Characteristic function of an SαS real vector $X^{(d)} = (X_1, \ldots, X_d)$ is given by:
$$\phi_X(\theta_1, \ldots, \theta_d) = \exp\{-\int_{S_d} |\theta_1 s_1 + \cdots + \theta_d s_d|^\alpha d\Gamma_{X^{(d)}}(s_1, \ldots, s_d)\}$$
where $\Gamma_{X^{(d)}}$ **is a unique positive finite measure** on the unit sphere of $\mathbb{R}^d$
4. Complexe random variables and vectors: $X = X_1 + \imath . X_2$ est α-stable if and only iff the vector (X_1, X_2) is α-stable on $\mathbb{R}^2$. More generally a vector $(X_1, \ldots, X_d)$ with $X_j = X_j^1 + \imath . X_j^2$, is α-stable if and only if $(X_1^1, X_1^2, \ldots, X_d^1, X_d^2)$ is α-stable vector on $\mathbb{R}^{2d}$.
5. A complexe SαS, $X = X_1 + \imath . X_2$ is said isotropic (rotationally invariant) if for any $\phi \in [0, 2\pi[$
$X \stackrel{d}{=} e^{\imath \phi} . X$.

α-stable processes, SαS random measures

1. A stochastic process $\xi = (\xi_t, t \in \mathbb{R})$ is symmetric if and only if its finite dimensional distributions are SαS vectors.
2. An SαS random measure is a random set function $d\xi : \mathcal{B}(\mathbb{R}) \longmapsto \mathbb{R}($ or $\mathbb{C})$ such that, for any Borel sets $A_1, \ldots, A_n$, the vector $(d\xi(A_1), \ldots, d\xi(A_n))$ is an $S\alpha$S random vector
3. A random measure $d\xi$ is said independently scattered if for any disjoint Borel sets $A_1, \ldots, A_n$ the variables $d\xi(A_1), \ldots, d\xi(A_n)$ are independents.

Dependence structure: the covariation

- Let $X = (X_1, X_2)$ jointly SαS vector with corresponding measure on the sphere Γ, the covariation of X_1 **on** X_2 is defined by :
$$[X_1, X_2]_\alpha = \int_{S_2} s_1.(s_2)^{<\alpha-1>} d\Gamma(s_1, s_2)$$
where $s^{<\beta>} = \mathbf{sign}(s).|s|^\beta$
- In case where $X = (X^1, X^2)$ is complex i.e. $X^1 = X_1^1 + \imath X_2^1$ and $X^2 = X_1^2 + \imath X_2^2$, then the covariation of X^1 on X^2 is :
$$[X^1, X^2]_\alpha = \int_{S_4} (s_1^1 + \imath s_2^1).(s_1^2 + \imath s_2^2)^{<\alpha-1>} d\Gamma_X(s_1^1, s_2^1, s_1^2, s_2^2)$$
and the notation $z^{<\beta>} = |z|^{\beta-1}\overline{z}$.

A useful result: For any SαS vector X on $\mathbb{R}^d$ with spectral measure Γ_X then,

$$\left[\sum_{i=1}^d a_i X_i, \sum_{i=1}^d b_i X_i\right]_\alpha = \int_{S_d} \left(\sum_{i=1}^d a_i s^i\right) . \left(\sum_{i=1}^d b_i s^i\right)^{<\alpha-1>} d\Gamma_X(s_1, \dots, s_d)$$

Properties of the covariation

1. Linearity with respect to the first component i.e. for any SαS vector (X_1, X_2, Y) we have, $[X_1 + X_2, Y]_\alpha = [X_1, Y]_\alpha + [X_2, Y]_\alpha$.
2. if X and Y are independent jointly SαS variables, then $[X, Y]_\alpha = 0$. the inverse is not true in general
3. the covariation is additive with respect to its second component, $[X, Y_1 + Y_2]_\alpha = [X, Y_1]_\alpha + [X, Y_2]_\alpha$ if Y_1 and Y_2 are independents.
4. for any real or complexe a and b, $[a.X, b.Y]_\alpha = ab^{<\alpha-1>}[X, Y]_\alpha$.
5. Let $X = (X_t)_t$ an SαS process and denote $l(X)$ the space of finite linear combinations of X. The application,

$$\|.\|_\alpha : \begin{array}{ll} l(X) & \longrightarrow \mathbb{R}^+ \\ Y & \longmapsto \qquad \|Y\|_\alpha \triangleq ([Y, Y]_\alpha)^{\frac{1}{\alpha}} \end{array}$$

   is a norm *covariation norm*. In this case $(l(X), \|.\|_\alpha)$ is a Banach space
6. In $(l(X), \|.\|_\alpha)$, the covariation is continuous moreover we have:

$$|[Z_1, Z_2]_\alpha - [Z_1, Z_3]_\alpha| \leq 2\|Z_1\|_\alpha.\|Z_2 - Z_3\|_\alpha^{\alpha-1}.$$

Second order processes - the stationary case

Let $r(t)$ be the covariance function of a second order stationary process X_t,

r(t) is positive definite

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i \overline{c}_j r(t_i - t_j) \geq 0$$

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$$\Longleftrightarrow$$

$$r(t) = \int_{-\infty}^{\infty} e^{\imath t\lambda} F(d\lambda)$$

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Second ordre - non stationary case

now the covariance is bivariate $r(s,t)$ and

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The bimeasure F is positive definite in the sense,

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$\Longrightarrow$

$F(A,B) = \text{cov}(\xi(A), \xi(B))$

F positive definite bi-measure

Cramer-Rao $\Longleftrightarrow$

$$X_t = \int_{-\infty}^{\infty} f(t,\lambda) d\xi(\lambda)$$

$$r(s,t) = \int\int f(s,\lambda)\overline{f(t,\lambda')}F(d\lambda, d\lambda')$$

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i.e. if $A \cap B = \emptyset$ **then** $d\xi(A)$ **inndep.** $d\xi(B)$

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In this case the covariation verify:

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and for the harmonisable case:

$$[X_s, X_t]_\alpha = \int e^{\imath\lambda(s-t)} \mu(d\lambda)$$

Spectral representation: extending the Camabanis-Miller representation

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- **Go further?** Apply this to harmonisable processes

The covariation additivity condition

Théorème

For the covariation to be additive with respect to its second variable i.e. $\forall i \in \{1, .., d\}, \forall \theta_1, ..., \theta_d \in \mathbb{C}$*:*

$$[X_i, \theta_1 X_1 + ... + \theta_d X_d]_\alpha = [X_i, \theta_1 X_1]_\alpha + ... + [X_i, \theta_d X_d]_\alpha,$$

it suffices that for all i, j *and* $k \in \{1, ..., d\}$

$$\forall \theta_1, ..., \theta_d \in \mathbb{R}, \qquad \frac{\partial^3 \phi}{\overline{\partial} \theta_i \overline{\partial} \theta_j \overline{\partial} \theta_k}(\theta_1, ...\theta_d) = 0. \tag{1}$$

where ϕ *is the Fourier transform of* Γ_X

Examples:

- Independent variables verify this conditions

$$\phi(\theta_1, \ldots, \theta_d) = a_1 \cos(\theta_1) + \cdots + a_1 \sin(\theta_1)$$

- A more general example:

$$\phi(\theta_1, ..., \theta_d) = \sum_{i \neq j = 1}^{d} \varphi_{i,j}(\theta_i, \theta_j)$$

where $\varphi_{i,j}$ are characteristic functions of finite measures on $\mathbb{S}_d$.

Construction of the bimeasure

Let us come back to the random measure $d\xi$

Definition

Condition (O):
we will say that it verifies the additivity condition if for all $n \geq 2$ and all disjoints Borelian sets $\{A_1, ..., A_n\}$ the $S\alpha S$ vector $(d\xi(A_1), ..., d\xi(A_n))$ verify the condition (1).

Let us now consider the set function F defined on $\mathcal{B}(\mathbf{R}) \times \mathcal{B}(\mathbf{R})$ by :

$$\mathbf{F}: \begin{array}{ccc} \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) & \longrightarrow & \mathbb{C} \\ (\mathbf{A}, \mathbf{B}) & \longmapsto & [\mathbf{d}\xi(\mathbf{A}), \mathbf{d}\xi(\mathbf{B})]_\alpha \end{array} \tag{2}$$

F is additive with respect to its two variables:it is a bimeasure.
For the second variable, if $\mathbf{B_1}$ and $\mathbf{B_2}$ two distinct Borel Sets,

$$\mathbf{F}(\mathbf{A}, \mathbf{B_1} \cup \mathbf{B_2}) = [\mathbf{d}\xi(\mathbf{A}), \mathbf{d}\xi(\mathbf{B_1} \cup \mathbf{B_2})]_\alpha = [\mathbf{d}\xi(\mathbf{A}), \mathbf{d}\xi(\mathbf{B_1}) + \mathbf{d}\xi(\mathbf{B_2})]_\alpha \tag{3}$$

Since $\mathbf{d}\xi$ satisfy the condition (O) then,

$$[\mathbf{d}\xi(\mathbf{A}), \mathbf{d}\xi(\mathbf{B_1}) + \mathbf{d}\xi(\mathbf{B_2})]_\alpha = [\mathbf{d}\xi(\mathbf{A}), \mathbf{d}\xi(\mathbf{B_1})]_\alpha + [\mathbf{d}\xi(\mathbf{A}), \mathbf{d}\xi(\mathbf{B_2})]_\alpha = \mathbf{F}(\mathbf{A}, \mathbf{B_1}) + \mathbf{F}(\mathbf{A}, \mathbf{B_2}).$$

the bimeasure $\mathbf{F}$ defined in (2) verifies a similar positive definitness property: for all complex $\mathbf{z_1}, \dots, \mathbf{z_n}$ and for all distinct Borel sets $\mathbf{A_1}, \dots, \mathbf{A_n}$, we have:

$$\sum_{\mathbf{i=1}}^{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{z_i}(\mathbf{z_j})^{<\alpha-1>} \mathbf{F}(\mathbf{A_i}, \mathbf{A_j}) \geq 0. \tag{4}$$

the proof of this property is easy. It suffices to use the condition (O). Indeed,

$$\begin{aligned}
\sum_{\mathbf{i=1}}^{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{z_i}(\mathbf{z_j})^{<\alpha-1>} \mathbf{F}(\mathbf{A_i}, \mathbf{A_j}) &= \sum_{\mathbf{i=1}}^{\mathbf{n}} \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{z_i}(\mathbf{z_j})^{<\alpha-1>} [\mathbf{d}\xi(\mathbf{A_i}), \mathbf{d}\xi(\mathbf{A_j})]_\alpha \\
&= \left[\sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{z_i} \mathbf{d}\xi(\mathbf{A_i}), \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{z_i} \mathbf{d}\xi(\mathbf{A_i}) \right]_\alpha \\
&= \left\| \sum_{\mathbf{i=1}}^{\mathbf{n}} \mathbf{z_i} \mathbf{d}\xi(\mathbf{A_i}) \right\|_\alpha^\alpha \geq 0
\end{aligned}$$

Some theoritical aspects...

Let's consider $\nu : \mathbf{A} \longmapsto \nu(\mathbf{A}) = \mathbb{E}(\mathbf{v}(\mathbf{d}\xi, \mathbf{A}))$ where $\mathbf{v}(\mathbf{d}\xi, \mathbf{A})$ is the total variation of the random measure $\mathbf{d}\xi$.
It is defined, for all borelian $\mathbf{A}$, by :

$$\mathbf{v}(\mathbf{d}\xi, \mathbf{A}) = \sup_{\mathbf{I} \text{ finite}} \left\{ \sum_{\mathbf{i} \in \mathbf{I}} |\mathbf{d}\xi(\mathbf{A_i})|, (\mathbf{A_i})_{\mathbf{i} \in \mathbf{I}} \text{ partition of } \mathbf{A} \right\} \tag{5}$$

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Since $\mathbf{d}\xi$ is with bounded variation then ν a bounded measure. With respect to this measure ν, we consider the norm of $\mathbf{L_1}(\nu)$ of a complex function $\mathbf{f}$ defined by,

$$\mathbf{N}(\mathbf{f}) = \int_{\mathbb{R}} |\mathbf{f}| \mathbf{d}\nu$$

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We denote by $\Lambda_\alpha(\mathbf{d}\xi)$ the completion, with respect to the norm $\mathbf{N}$

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For purpose of constructing the covariation spectral representation we need the next result that ensure the convergence theorems

Proposition

Suppose that $\mathbf{d}\xi$ *satisfy the condition (O), then we have the next properties:*

1. *For all* $\mathbf{A} \in \mathcal{B}(\mathbb{R})$ *we have ,* $\|\mathbf{d}\xi(\mathbf{A})\|_\alpha \leq \Psi_\alpha(1).\nu(\mathbf{A})$ *where* $\Psi_\alpha(\mathbf{p})$ *is equal to* $\frac{1}{\mathcal{S}_\alpha(\mathbf{p})}$ *when* ξ *is real and is equal to* $\frac{1}{\tilde{\mathcal{S}}_\alpha(\mathbf{p})}$ *in the complex case. The quantities* $\mathcal{S}_\alpha(\mathbf{p})$ *et* $\tilde{\mathcal{S}}_\alpha(\mathbf{p})$ *depends only on* α *and* $\mathbf{p}$*.*
2. *Let* $\mathbf{B} \in \mathcal{B}(\mathbb{R})$ *a fix Borel set. If* $\mathbf{A}$ *is set verifying* $\nu(\mathbf{A}) = 0$ *then the total variation variation of the complex measure* $\mathbf{F_B}$ *in* $\mathbf{A}$ *is null , that is* $\mathbf{v}(\mathbf{F_B}, \mathbf{A}) = 0$*. This result is also true for the measure* $\tilde{\mathbf{F}}_\mathbf{A}(\mathbf{B})$ *and B fixed.*
3. *Let* $\mathbf{B} \in \mathcal{B}(\mathbb{R})$ *be a fix Borel set, then for all bounded function* $\mathbf{f} \in \Lambda_\alpha(\mathbf{d}\xi)$ *we have the inequality :*

$$|\int_{\mathbb{R}} \mathbf{f}\mathbf{d}\mathbf{F_B}| \leq \Psi_\alpha(1).\|\mathbf{d}\xi(\mathbf{B})\|_\alpha^{\alpha-1} \int_{\mathbb{R}} |\mathbf{f}|\mathbf{d}\nu \qquad (6)$$

4. *Let* $\mathbf{f} \in \Lambda_\alpha(\mathbf{d}\xi)$ *be a fixed bounded function and note* $\mathbf{G}(\mathbf{B}) = \tilde{\mathbf{I}}(\mathbf{f}, \mathbf{B})$*. We have the next implication:* $(\nu(\mathbf{B}) = 0$ *implies* $\mathbf{v}(\mathbf{G}, \mathbf{B}) = 0)$*.*

Proposition

Let $\mathbf{f}$ *and* $\mathbf{g}$ *Two bonded function in* $\Lambda_\alpha(\mathbf{d}\xi)$*. The covariation of the stochastic integral* $\int_{\mathbb{R}} \mathbf{fd}\xi$ *on* $\int_{\mathbb{R}} \mathbf{gd}\xi$ *is given by:*

$$\left[\int_{\mathbb{R}} \mathbf{fd}\xi, \int_{\mathbb{R}} \mathbf{gd}\xi\right]_\alpha = \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbf{f}(\lambda)\left(\mathbf{g}(\lambda')\right)^{<\alpha-1>} \mathbf{F}(\mathbf{d}\lambda, \mathbf{d}\lambda'). \tag{7}$$

Proposition

Suppose that ξ *is a real or complex isotrope symmetric* α*-stable process. Then the bimeasure* $\mathbf{F}$ *defined in (2) is the unique bimeasure characterizing the process* $\mathbf{X}$ *and verifying the integral representation (7).*