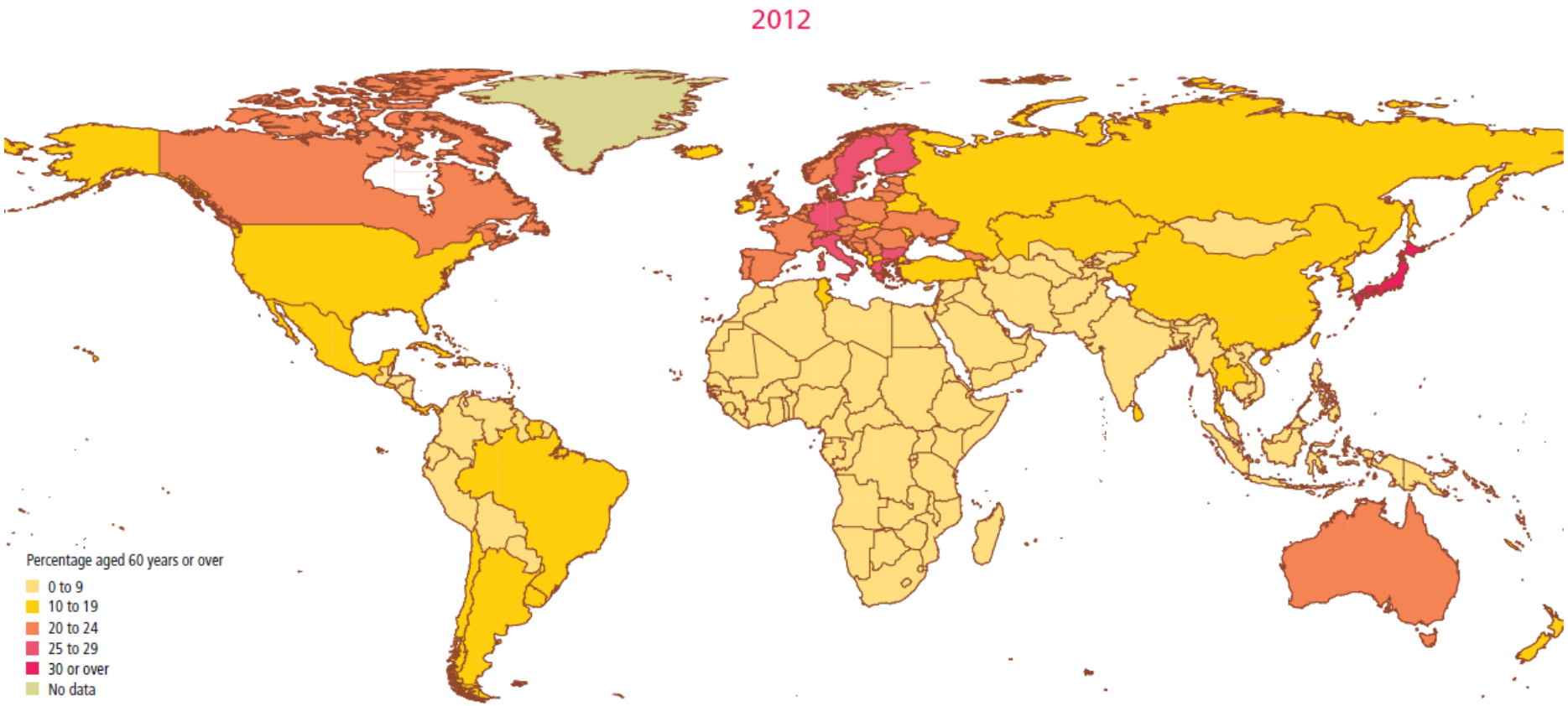
Modelling Annuity Portfolios and Longevity Risk with Extended CreditRisk$^+$

Pavel Shevchenko
Commonwealth Scientific and Industrial Research Organisation (CSIRO) Australia

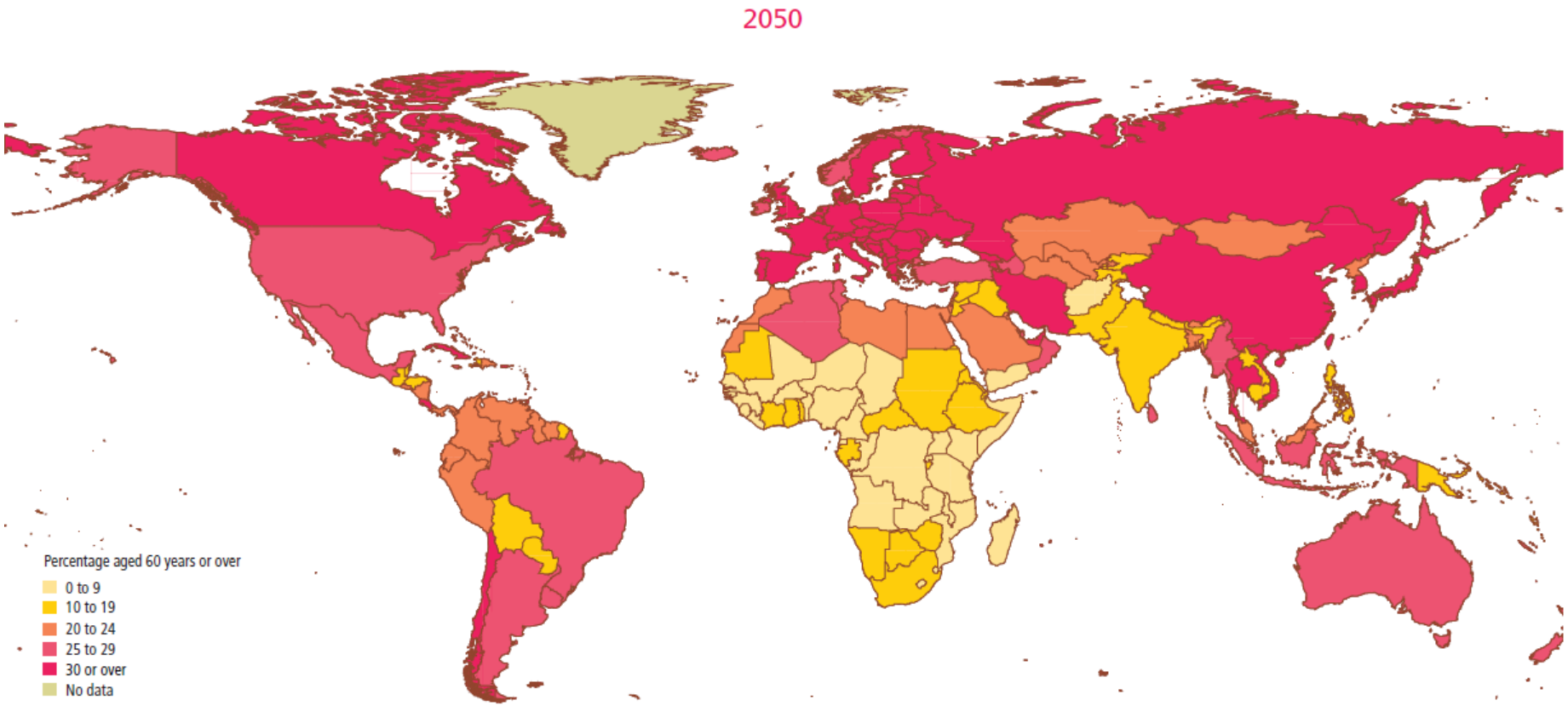
joint work with Uwe Schmock and Jonas Hirz, Vienna TU

International Workshop on Spatial and Temporal Modeling from Statistical, Machine Learning and Engineering perspectives (STM 2015)
and
International Workshop on Complex Systems Modeling and Estimation Challenges in Big Data (CSM 2015)
13-17 July 2015

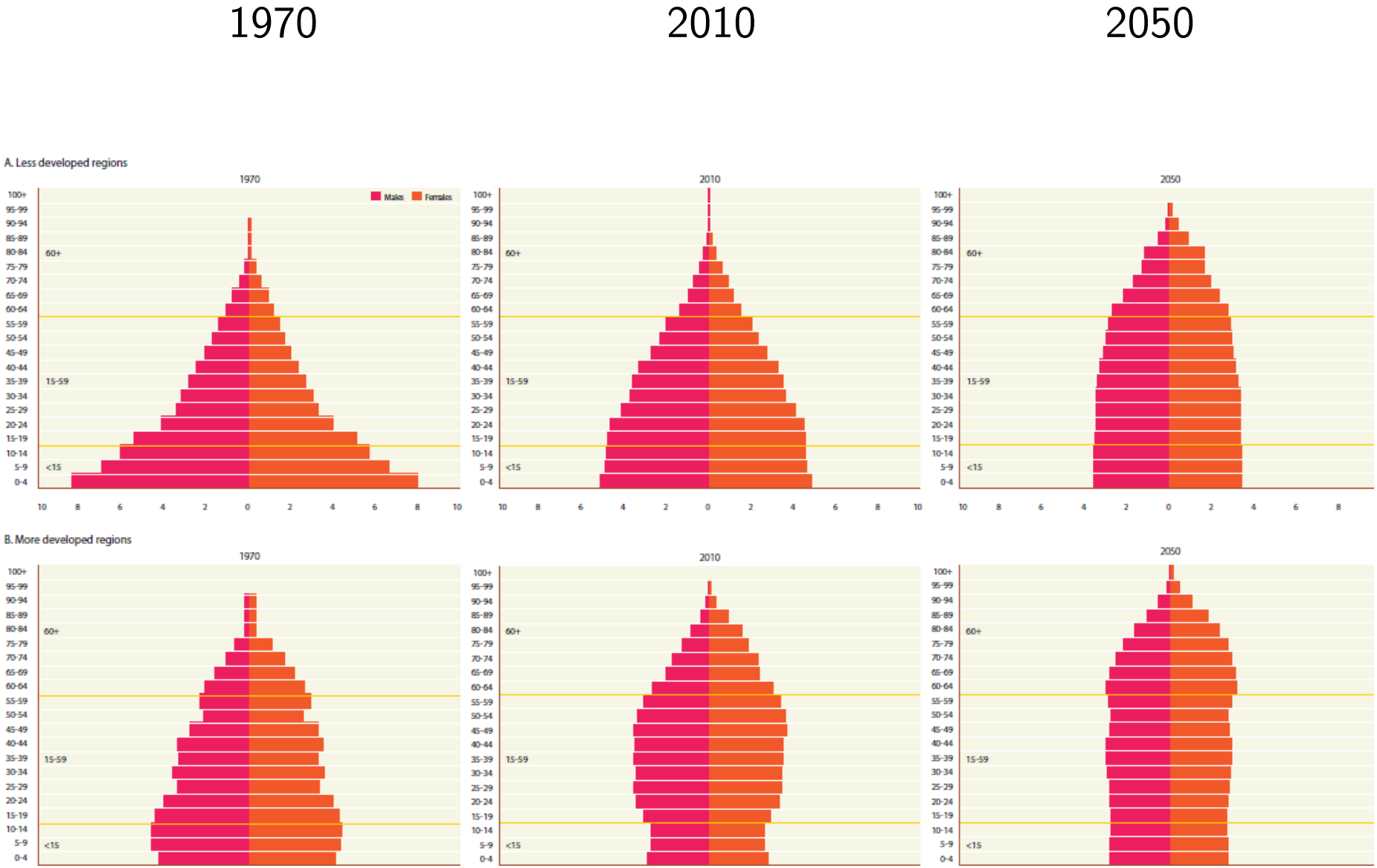
Population Aging - United Nations Data (2012): percentage aged 60 years or over, 2012 vs 2050 forecast



Population Aging - United Nations Data (2012): percentage aged 60 years or over, 2012 vs 2050 forecast



Age distribution of the world's population



- **The world population is getting older fast!** Life expectancy 78 years (2010-2015) and 83 years by 2045-2050 in developed regions. Older persons will outnumber children (0-14 years) soon.
- **Pressure on the government budgets**: in Australia the total age/service pension payments are $34.8 billion in 2011-12.
- **Australian superannuation industry is large (and getting larger)**. $1.6 tn in assets under management, it is now greater than the capitalization of the ASX, and greater than the combined deposits of all Australian banks; exceeds the size of domestic GDP.
- **Longevity risk** (potential risk attached to the increasing life expectancy of pensioners and policy holders "outliving one's savings") – potential solutions for retirees: purchase "peace of mind" - e.g. annuities; access home equity - reverse mortgage; rely on government pension; Continue Working!
- **CSIRO-Monash superannuation research cluster 2013**-: retirement products (variable annuities with guarantee features), depletion rates of super balance, longevity risk, life-cycle utility model, superannuation and economy.

Outline

- We develop a model to derive loss distributions of annuity portfolios over one period.
- The model is based on extended CreditRisk$^+$.
- There exists a numerically stable and fast algorithm to derive loss distributions and risk measures exactly.
- Based on publicly available data we provide estimation procedures, including MCMC.
- The model can also be applied to model life tables and mortality forecasts.
- Stress scenarios can also be tested.
- Setup to model new insurance contracts.

This talk is based on draft papers:
Hirz, Schmock and Shevchenko (2015) available on http://arxiv.org/abs/1505.04757
Shevchenko, Hirz and Schmock (2015) to appear in MODSIM 2015 proceedings.

Mortality modelling

- **Deterministic survival models** for $\Pr(T_x > t) = \exp(-\int_0^t \mu_{x+s} ds)$, T_x is remaining lifetime of a person aged x and μ_s is mortality intensity: **Gompertz (1825)** $\mu_x = Bc^x$, **Makeham (1860)** $\mu_x = A + Bc^x$, **Perks (1932)** $\mu_x = \frac{A+Be^{\gamma x}}{1+Ce^{\gamma x}}$; **Thatcher (1999)** $A + \frac{Be^{\gamma x}}{1+Be^{\gamma x}}$, etc.
- **Subjective opinions of experts**
- **Stochastic modelling** is more recent development (state-space models, GLM, etc.): benchmark method is **Lee–Carter model** (1992), given the number of living people $m_{\mathrm{a,g}}(t)$ as well as annual deaths $n_{\mathrm{a,g}}(t)$, for age a, gender g and years $t \in \{1, \ldots, T\}$, the death rates are modelled as

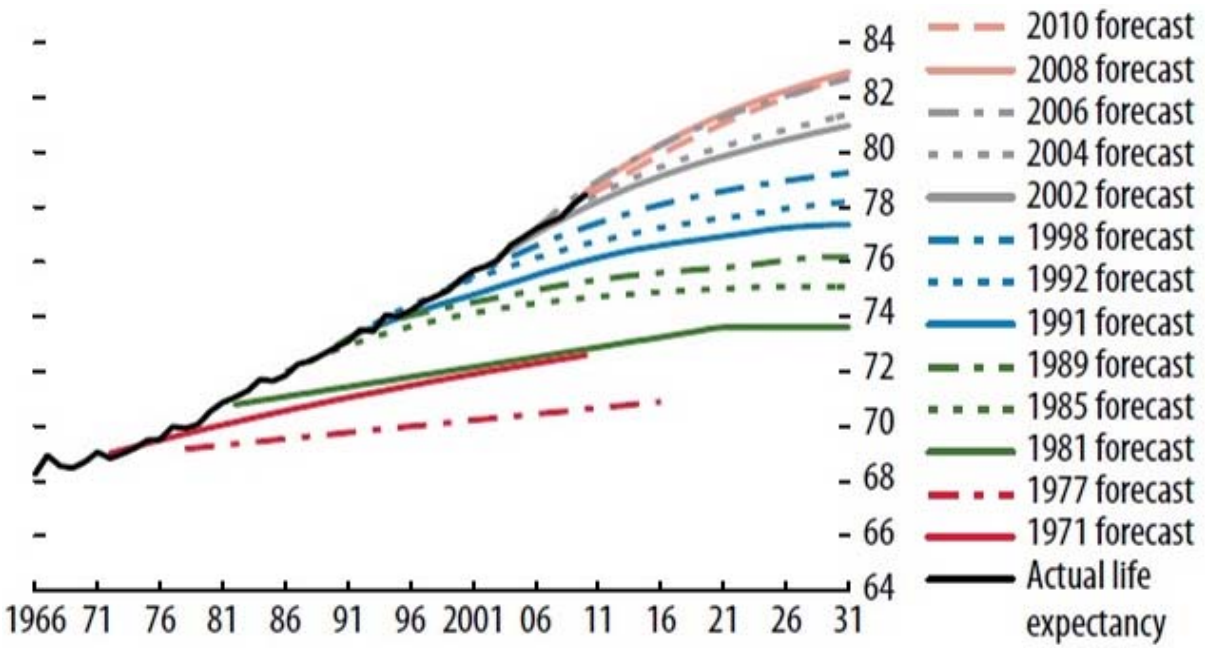
$$\log \hat{q}_{\mathrm{a,g}}(t) = \log \frac{n_{\mathrm{a,g}}(t)}{m_{\mathrm{a,g}}(t)} = a_{\mathrm{a,g}} + \kappa_t \, b_{\mathrm{a,g}} + \varepsilon_{\mathrm{a,g},t} \,, \quad t \in \{1, \ldots, T\} \,,$$

  with independent normal error terms $\varepsilon_{\mathrm{a,g},t}$.
- **Life Tables** – point estimators for death probabilities

Observation I

When applying life tables to annuities, death probabilities have to be reduced by minimal risk margins to account for longevity (e.g. DAV in Germany):

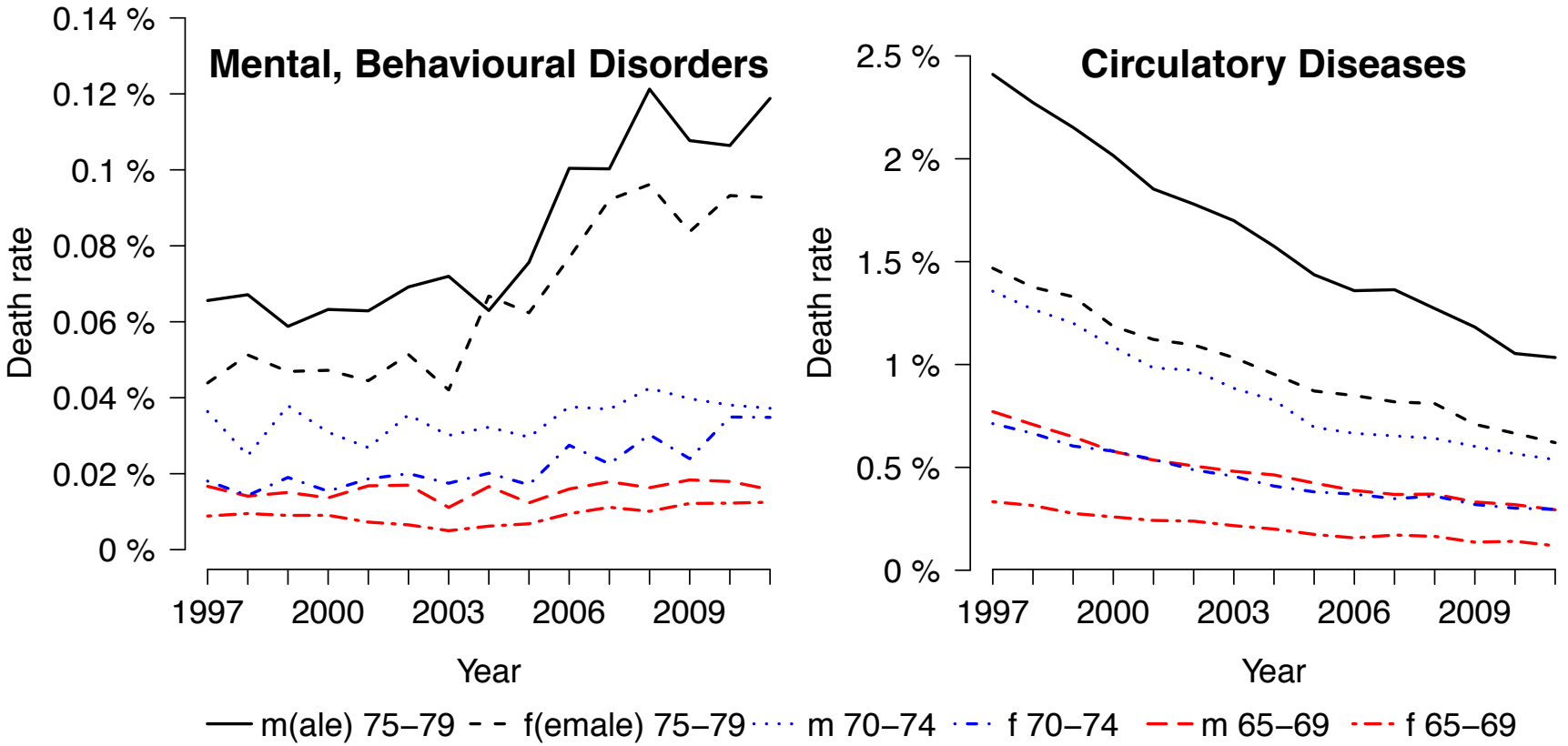
- Mortality trends: For example, Lee–Carter model.
- $\sim 7\%$: Statistical fluctuation.
- 10%: Parameter risk, structural differences.
- 15%: Selection risk.



UK, projected life expectancy at birth for males 1966-2031. Office of National Statistics

Observation II

Australian mortality rates due to different death causes show significant patterns, also on a short-term scale (1997-2011).



Goal

Develop a model which derives **loss distributions** of annuity portfolios over one period which

- takes into account some of the risks mentioned before to account for **longevity**,
- accounts for changes in rates of different **death causes**,
- accounts for dependence between policyholders,
- has a potentially **short runtime** (not Monte Carlo),
- can model **any kind of annuity** (index-linked, variable annuities),
- has the feature of **stress testing**.
- forecast of death rates and death causes

Extended CreditRisk$^+$ (ECRP)

Collective risk model *extended CreditRisk*$^+$ (see Schmock (2014), short ECRP, and CRP (1997)) is able to cover all those attributes, if

default is treated as **death** .

We know:

- ECRP allows an explicit calculation of the loss distribution via a stable and fast algorithm.
- ECRP can be applied to any kind of annuity (index-linked, variable annuities).
- ECRP allows flexible handling of dependence through common stochastic risk factors.

Notation and setup

- **Policyholders** $1, \ldots, m$.
- **Death indicators** $N_1, \ldots, N_m \in \mathbb{N}_0$ (random variables) where $\{N_i = 0\}$ indicates 'no death'.
- Independent stochastic or deterministic **annuity payments** $X_1, \ldots, X_m \in \mathbb{N}_0$ (may be multi-dimensional including discounted actuarial reserve and different lines of business) and annuities which need not be paid in the case of death $Y_1, \ldots, Y_m \in \mathbb{N}_0$, mutually indep. and indep. of $N_1, \ldots, N_m$.

Total portfolio loss

For i.i.d. copies $\{Y_{i,j}\}_{j \in \mathbb{N}}$ of Y_i, for $i \in \{1, \ldots, m\}$, derive

$$L := \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} \sum_{j=1}^{N_i} Y_{i,j} \, .$$

Death indicator

What assumptions on death indicators N_i in

$$S := \sum_{i=1}^{m} \sum_{j=1}^{N_i} Y_{i,j} \, ?$$

- In reality, (N_i) are **Bernoulli** distributed: Monte Carlo.
- If (N_i) are independently **Poisson** distributed with mean (λ_i), then Panjer's recursion can be applied, i.e., for $\lambda := \sum_{i=1}^{m} \lambda_i$, $q_\nu := \sum_{i=1}^{m} (\lambda_i/\lambda) \mathbb{P}(Y_i = \nu)$, and $\mathbb{P}(S = 0) = \exp(-\lambda)$

$$\mathbb{P}(S = s) = \frac{\lambda}{s} \sum_{\nu=1}^{s} \nu \, q_\nu \, \mathbb{P}(S = s - \nu), \quad s \in \mathbb{N}_0.$$

- If (N_i) are **compound Poisson** distributed, then Panjer's recursion can still be applied in some cases as in (extended) CreditRisk$^+$.

Note on basic Panjer recursion

Calculation of distribution of $Z = X_1 + \ldots + X_N$, where N is random, is classical problem of risk theory

- Define $f_k = \Pr[X_i = k\delta], \quad p_k = \Pr[N = k], \quad h_k = \Pr[Z = k\delta]$, with $f_0 = 0$ and $k = 0, 1, \ldots$. Then

$$\begin{aligned} h_n &= \sum_{k=1}^{n} p_k f_n^{(k)*}, \quad n \geq 1, \\ h_0 &= \Pr[Z = 0] = \Pr[N = 0] = p_0, \end{aligned}$$

  where $f_n^{(k)*} = \sum_{i=0}^{n} f_{n-i}^{(k-1)*} f_i$ with $f_0^{(0)*} = 1$ and $f_n^{(0)*} = 0$ if $n \geq 1$.
- The number of operations to calculate $h_0, h_1, \ldots, h_n$ using convolutions explicitly is of the order of n^3.
- If the frequency N belongs to the so-called Panjer classes, calculation is reduced to a simple recursion introduced by H. Panjer in 1981 and referred to as Panjer recursion, that required $O(n^2)$ operations.

Note on basic Panjer recursion

Theorem (Panjer recursion)

If the frequency probability mass function p_n, $n = 0, 1, \ldots$ *satisfies*

$$p_n = \left(a + \frac{b}{n}\right) p_{n-1}, \quad \text{for} \quad n \geq 1 \quad \text{and } a, b \in \mathbb{R},$$

then it is said to be in Panjer class $(a, b, 0)$ *and the compound distribution satisfies the recursion*

$$\begin{aligned} h_n &= \frac{1}{1 - af_0} \sum_{j=1}^{n} \left(a + \frac{bj}{n}\right) f_j h_{n-j}, \quad n \geq 1, \\ h_0 &= \sum_{k=0}^{\infty} (f_0)^k p_k. \end{aligned}$$

- Poisson, Binomial, Negative Binomial belong to Panjer class $(a, b, 0)$.
- There are several extensions to Panjer class and Panjer recursion; see Cruz, Peters and Shevchenko (2015) and Peters and Shevchenko (2015).

Multiple deaths

Multiple deaths of a single policyholder can occur when using (compound) Poisson distributed deaths, but:

Multiple deaths are not a major issue

- Since annual death probabilities are small for most ages, multiple deaths are unlikely.
- Multiple deaths is not a major issue for longevity risk modelling.
- Approximations using Poisson sums are justified by Poisson approximation and generalisations of this result (Vellaisamy and Chaudhuri (1996)).
- With proper scaling, we get accurate results (next example).

Illustrative example

- $m = 1\,000$ policyholders with annual death probability $q = 0.05$.
- For policyholder i, let X_i be LogNormal with $\mu = 4$ and $\sigma = 0.5$. Let $U_i \sim U(0, 1]$ and define $Y_i := X_i U_i$.

Using 10 000 simulations derive $S := \sum_{i=1}^{m} \sum_{j=1}^{N_i} Y_{i,j}$ where $Y_{i,j} \sim Y_i$, for N_i being Poisson as well as Bernoulli distributed, both with $\mathbb{P}(N_i = 0) = 1 - q$, $i \in \{1, \ldots, m\}$.

		Bernoulli	Poisson
VaR(S)	0.01	1 007.87	1 005.16
	0.05	1 174.09	1 170.17
	0.15	1 325.84	1 324.91
	0.99	2 333.72	2 373.00

Annuity model using extended CreditRisk$^+$

Annuity model with risk factors

For all policyholders $i \in \{1, \ldots, m\}$:

- **Annual death probability** q_i^* and set $q_i := -\log(1 - q_i^*)$.
- **Risk factors** $\Lambda_1, \ldots, \Lambda_K$ are independent and have gamma distributions with mean 1 and variances $\beta_1, \ldots, \beta_K$.
- Death indicators are split up $N_i = N_{i,0} + N_{i,1} + \cdots + N_{i,K}$ due to different risk factors (**death causes**) with corresponding **weights** $w_{i,0}, \ldots, w_{i,K} \geq 0$ such that $w_{i,0} + \cdots + w_{i,K} = 1$.
- $N_{i,0}$ is independent of everything else, $\mathcal{L}(N_{i,0}) = \mathrm{Poi}(q_i\, w_{i,0})$.
- $(N_{i,k})_{i,k}$ are conditionally independent given $\Lambda_1, \ldots, \Lambda_K$ and they have a **compound Poisson** distribution

$$\mathcal{L}(N_{i,k} \,|\, \Lambda_1, \ldots, \Lambda_K) = \mathcal{L}(N_{i,k} \,|\, \Lambda_k) = \mathrm{Poi}(q_i\, w_{i,k}\, \Lambda_k)$$

Interpretation and comments on the annuity model

- **Risk factors** $\Lambda_1, \ldots, \Lambda_K$ represent **causes of death** such as neoplasms, cardiovascular diseases or idiosyncratic components. The variation in this risk factors represents unexpected improvement in medication or outbursts of epidemics, etc.
- E.g., a low value of the risk factor for neoplasms Λ_k reduces the Poisson intensity in $\mathrm{Poi}(q_i w_{i,k} \Lambda_k)$ and implies reduced death probability which may be the case if a new cancer treatment is available.
- The weights $w_{i,k}$ indicate how vulnerable policyholder i is to risk factor Λ_k.
- General case of extended CreditRisk+ can be used to model **risk groups with simultaneous deaths** of policyholders in the group (e.g. couple dies in a car crash, people living near a vulcan, virus outbreaks), **further dependence (negative and positive) between death causes** can be introduced via linear structure $\Lambda_c = a_{c,0} + a_{c,1} R_1 + \cdots + a_{c,K} R_K$ where $R_k \sim Gamma(\alpha_k, \beta_k)$.

Moments of $N_{i,k}$

Given the annuity model with K non-idiosyncratic risk factors, let $k \in \{1, \ldots, K\}$ and consider policyholder $i \in \{1, \ldots, m\}$. Then, for the number of deaths $N_{i,k}$ due to risk factor Λ_k we have

$$\mathbb{E}[N_{i,k}] = \mathbb{E}[\mathbb{E}[N_{i,k}\,|\,\Lambda_k]] = \mathbb{E}[q_i\, w_{i,k}\, \Lambda_k] = q_i\, w_{i,k}\,,$$

$$\begin{aligned}\mathrm{Var}(N_{i,k}) &= \mathbb{E}[\mathrm{Var}(N_{i,k}\,|\,\Lambda_k)] + \mathrm{Var}(\mathbb{E}[N_{i,k}\,|\,\Lambda_k]) \\ &= q_i\, w_{i,k}(1 + q_i\, w_{i,k}\beta_k)\,.\end{aligned}$$

Analogously, for all $i, j \in \{1, \ldots, m\}$ with $i \neq j$,

$$\begin{aligned}\mathrm{Cov}(N_{i,k}, N_{j,k}) &= \mathbb{E}[\mathrm{Cov}(N_{i,k}, N_{j,k}\,|\,\Lambda_k)] + \mathrm{Cov}(\mathbb{E}[N_{i,k}\,|\,\Lambda_k]\,, \mathbb{E}[N_{j,k}\,|\,\Lambda_k]) \\ &= q_i\, q_j\, w_{i,k}\, w_{j,k}\beta_k\,.\end{aligned}$$

Algorithm for (extended) CreditRisk$^+$

For $S = \sum_{i=1}^{m} \sum_{j=1}^{N_i} Y_{i,j}$ the algorithm is given by

$$\mathbb{P}(S = \nu) = \frac{\lambda}{\nu} \sum_{n=1}^{\nu} n c_n \mathbb{P}(S = \nu - n), \quad \nu \in \mathbb{N},$$

where $\mathbb{P}(S = 0) = \exp\big(\lambda(c_0 - 1)\big)$ with $\lambda, c_0 \in \mathbb{R}$ and

$$c_\nu = f(b_{1,\nu}, \ldots, b_{K,\nu}), \quad \nu \in \mathbb{N}_0,$$

where, for all $k \in \{1, \ldots, K\}$, $b_{k,0} \in \mathbb{R}$ and

$$b_{k,\nu} = g_\nu(b_{k,1}, \ldots, b_{k,\nu-1}), \quad \nu \in \mathbb{N}_0,$$

with some functions $g_1, g_2, \ldots, f$.

Idea of proof: Deriving the probability-generating function of S for all $z \in \mathbb{C}$ with $\|z\|_\infty \leq 1$ gives

$$\mathbb{E}\big[z^S\big] = \sum_{\nu \in \mathbb{N}_0} \mathbb{P}(S = \nu) z^\nu = \exp\big(\lambda(\tilde{\varphi}(z) - 1)\big), \tag{1}$$

where $\tilde{\varphi}(z) = \sum_{\nu \in \mathbb{N}_0} c_\nu z^\nu$. The form of the probability-generating function implies that S is a Poisson sum, see Schmock (2014).

Available data

- Historical data of annual number of deaths $n_{\mathrm{a,g},k}(t) \in \mathbb{N}_0$ categorised by age $\mathrm{a} \in \{1, \ldots, \mathrm{A}\}$, gender $g \in \{f, m\}$ and death cause $k \in \{0, \ldots, K\}$ for years $t \in \{1, \ldots, T\}$.
- For Australia: Long-term data for 18 age groups, both genders with 19 death causes available.
- Corresponding historical population counts $m_{\mathrm{a,g}}(t)$.

Data and model linkage

$n_{\mathrm{a,g},k}(t)$ correspond to realisations of the random variable

$$N_{\mathrm{a,g},k}(t) := \sum_{i=1}^{m_{\mathrm{a,g}}(t)} N_{i,k}(t)\,,$$

Assumptions for estimation

Simplifying assumptions for consistent estimation

Additionally assume the following:

- Weights and death probabilities are the same within each age category and gender.
- Risk factor variances $\beta_1, \ldots, \beta_K$ are constant over the years.
- All random variables are independent for different points in time.
- Over short periods
  trends in death probabilities take the form
  $$\log q_{\mathrm{a,g}}(t) = a_{\mathrm{a,g}} + (T - t)\, b_{\mathrm{a,g}}\,,$$
  and **trends in weights**
  $$w_{\mathrm{a,g},k}(t) = c_{\mathrm{a,g},k} + (T - t)\, d_{\mathrm{a,g},k}\,.$$

Modelling trends over long time

- **Death probabilities:** $q_{a,g}(t) = F^{\mathrm{Lap}}\left(\alpha_{a,g} + \beta_{a,g}\,\mathcal{T}_{\zeta_{a,g},\eta_{a,g}}(t)\right)$, where $\alpha_{a,g}, \beta_{a,g}, \zeta_{a,g} \in \mathbb{R}$ and $\eta_{a,g} \in (0,\infty)$
- **Weights:** $w_{a,g,k}(t) = \dfrac{\exp\left(u_{a,g}+v_{a,g}\,\mathcal{T}_{\phi_k,\psi_k}(t)\right)}{\sum_{j=0}^{K}\exp\left(u_{a,g,j}+v_{a,g,j}\mathcal{T}_{\phi_j,\psi_j}(t)\right)}$, with $u_{a,g,0}, v_{a,g,0}, \phi_0, \ldots, u_{a,g,K}, v_{a,g,K}, \phi_K \in \mathbb{R}$, $\psi_0, \ldots, \psi_K \in (0,\infty)$.

- $F^{\mathrm{Lap}}(x)$ is **Laplace distribution** with mean one and variance two

$$F^{\mathrm{Lap}}(x) = \frac{1}{2} + \frac{1}{2}\mathrm{sign}(x)\left(1 - \exp(-|x|)\right), \quad x \in \mathbb{R},$$

  For $x < 0$, $F^{\mathrm{Lap}}(x) = \exp(x)/2$.

- **Trend reduction** with parameters $(\zeta, \eta) \in \mathbb{R} \times (0,\infty)$ is given by

$$\mathcal{T}_{\zeta,\eta}(t) = \frac{1}{\eta}\arctan(\zeta + \eta t), \quad t \in \mathbb{R}.$$

  Note, $\lim_{x\to\pm\infty}\arctan(x) = \pm\frac{\pi}{2}$ and η gives the inverse of time when an initial trend is halved.

Estimation procedures

Using these assumptions, we can develop several estimation approaches:

- **Matching of moments**: Easy to calculate and reasonably accurate.
- **Maximum a posteriori**: MAP-function is given explicitly but numerical deterministic optimisation is problematic (362 parameters). Risk factor realisations can be estimated (stress testing) and handy approximations can be derived.
- **Maximum likelihood**: ML-function is given explicitly but numerical optimisation is hard (**362 parameters**).
- **Markov chain Monte Carlo**: Based on likelihood function, switching to a Bayesian setting, parameters can be estimated accurately. This approach is slow but provides the feature of posterior densities of parameters.

Estimation procedures

Lemma (The maximum a posteriori approach)

For $k \in \{1, \ldots, K\}$ and $t \in \{1, \ldots, T\}$, given the posterior density $\pi(\boldsymbol{\beta}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \boldsymbol{\lambda} | \mathbf{n}) \propto \pi(\mathbf{n} | \boldsymbol{\beta}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \boldsymbol{\lambda}) \pi(\boldsymbol{\lambda} | \boldsymbol{\beta}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) \pi(\boldsymbol{\beta}, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d})$ *from the maximum a posteriori approach, we get by taking partial derivatives*

$$\hat{\lambda}_k(t) = \frac{1/\hat{\beta}_k^{\mathrm{MAP}} - 1 + \sum_{\mathrm{a}=1}^{\mathrm{A}} \sum_{\mathrm{g} \in \{\mathrm{f},\mathrm{m}\}} n_{\mathrm{a},\mathrm{g},k}(t)}{1/\hat{\beta}_k^{\mathrm{MAP}} + \sum_{\mathrm{a}=1}^{\mathrm{A}} \sum_{\mathrm{g} \in \{\mathrm{f},\mathrm{m}\}} \rho_{\mathrm{a},\mathrm{g},k}(t)}$$

as well as

$$\log \hat{\beta}_k^{\mathrm{MAP}} + \frac{\Gamma'\left(1/\hat{\beta}_k^{\mathrm{MAP}}\right)}{\Gamma\left(1/\hat{\beta}_k^{\mathrm{MAP}}\right)} = \frac{1}{T} \sum_{t=1}^{T} \left(1 + \log \hat{\lambda}_k(t) - \hat{\lambda}_k(t)\right),$$

where for given $\hat{\lambda}_k(1), \ldots, \hat{\lambda}_k(T) > 0$, the latter equation has a unique positive solution.

Likelihood function

$$\ell_{\mathbf{n}} = \prod_{t=1}^{T} \left(\left(\prod_{\mathrm{a}=1}^{\mathrm{A}} \prod_{\mathrm{g}\in\{\mathrm{f},\mathrm{m}\}} \frac{e^{-\rho_{\mathrm{a},\mathrm{g},0}(t)}\, \rho_{\mathrm{a},\mathrm{g},0}(t)^{n_{\mathrm{a},\mathrm{g},0}(t)}}{n_{\mathrm{a},\mathrm{g},k}(t)!} \right) \right.$$
$$\times \prod_{k=1}^{K} \left(\frac{\Gamma(1/\beta_k + n_k(t))}{\Gamma(1/\beta_k)\, \beta_k^{1/\beta_k}\, (1/\beta_k + \rho_k(t))^{1/\beta_k + n_k(t)}} \right.$$
$$\left. \left. \times \prod_{\mathrm{a}=1}^{\mathrm{A}} \prod_{\mathrm{g}\in\{\mathrm{f},\mathrm{m}\}} \frac{\rho_{\mathrm{a},\mathrm{g},k}(t)^{n_{\mathrm{a},\mathrm{g},k}(t)}}{n_{\mathrm{a},\mathrm{g},k}(t)!} \right) \right)$$

where, $n_k(t) := \sum_{\mathrm{a}=1}^{\mathrm{A}} \sum_{\mathrm{g}\in\{\mathrm{f},\mathrm{m}\}} n_{\mathrm{a},\mathrm{g},k}(t)$, as well as $\rho_{\mathrm{a},\mathrm{g},k}(t) := m_{\mathrm{a},\mathrm{g}}(t)\, q_{\mathrm{a},\mathrm{g}}(t)\, w_{\mathrm{a},\mathrm{g},k}(t)$ and $\rho_k(t) := \sum_{\mathrm{a}=1}^{\mathrm{A}} \sum_{\mathrm{g}\in\{\mathrm{f},\mathrm{m}\}} \rho_{\mathrm{a},\mathrm{g},k}(t)$.

Illustration example setup

- **Periods** $t \in \{1, \ldots, 10\}$.
- Two **age categories** $(\mathrm{a}_1, \mathrm{a}_2)$ with 10 000 policyholders each and one gender g.
- Annual **death probabilities** between 0.005 and 0.1.
- Two non-idiosyncratic **risk factors** Λ_1, Λ_2 with variances $\beta_1 = 0.05$ and $\beta_2 = 0.2$.
- **Weights** $w_{\mathrm{a}_1,\mathrm{g},1} = 0.1$, $w_{\mathrm{a}_2,\mathrm{g},1} = 0.2$, $w_{\mathrm{a}_1,\mathrm{g},2} = 0.3$ and $w_{\mathrm{a}_2,\mathrm{g},2} = 0.4$ which are assumed to be constant over time.

Number of deaths $n_{\mathrm{a},\mathrm{g},k}(t)$ are then generated via simulated risk factor realisations $(\lambda_1(t), \lambda_2(t))_{t \in \{1,\ldots,10\}}$ and simulation of Poisson distributions with parameters $p_{\mathrm{a}_i,\mathrm{g}}\, w_{\mathrm{a}_i,\mathrm{g},j}\, \lambda_j(t)$.

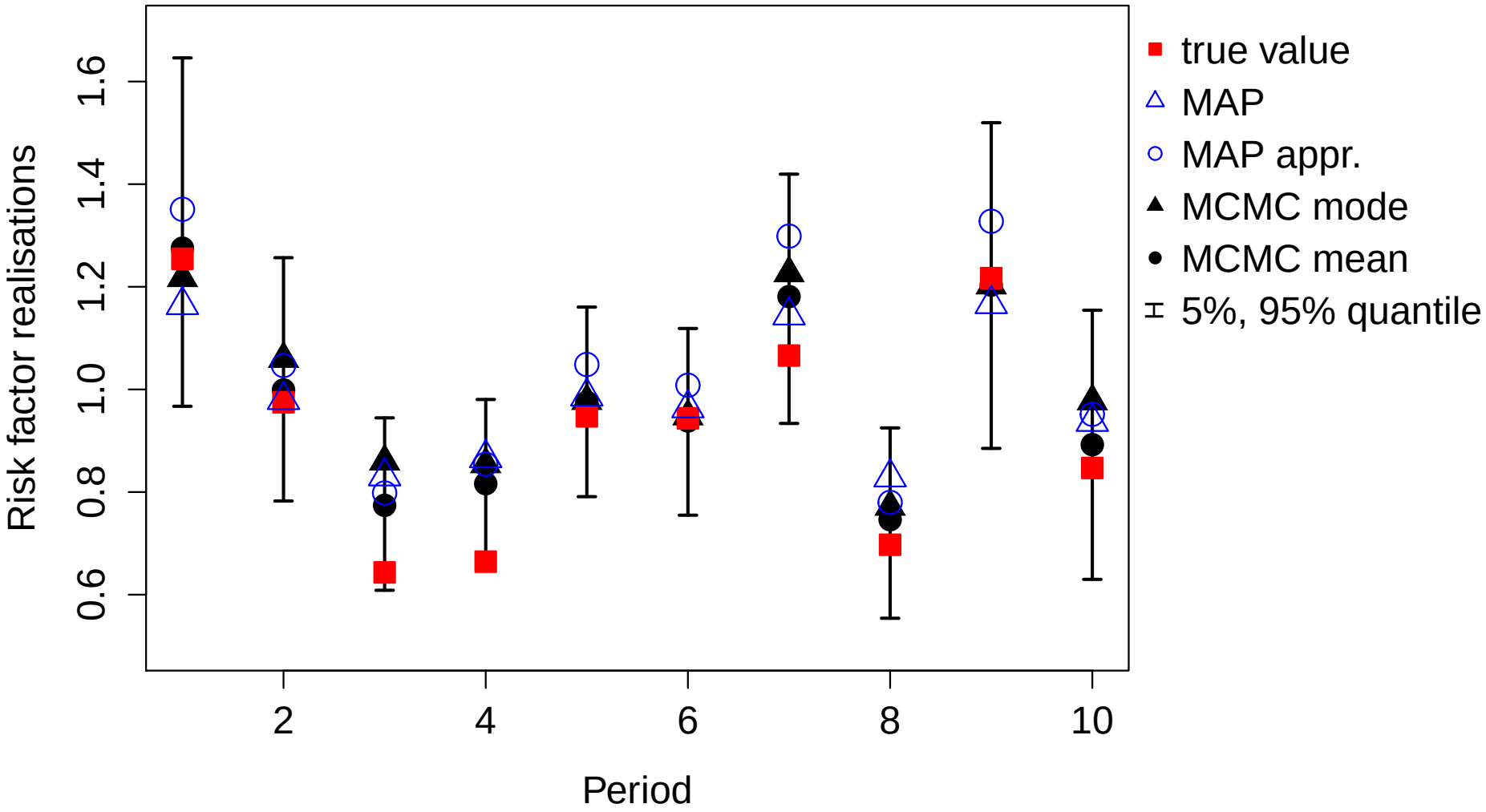
Estimation results

Using all direct estimation procedures as well as MCMC (Random walk Metropolis–Hastings within Gibbs) for the maximum a posteriori approach we get the following:

	β_1	β_2	$c_{a_2,g,1}$	$d_{a_2,g,1}$
true	0.050	0.200	0.200	0.000
MM	0.054	0.267	0.161	0.003
MAP	0.015	0.218	0.158	0.003
MLE	0.032	0.215	0.152	0.006
MAP MCMC single	β_1	β_2	$c_{a_2,g,1}$	$d_{a_2,g,1}$
mode	0.020	0.297	0.148	0.004
mean	0.084	0.377	0.164	0.004
5% quantile	0.025	0.152	0.125	-0.004
95% quantile	0.210	0.793	0.202	0.013
standard error	0.216%	0.441%	0.119%	0.027%

Estimation results: risk factor realisations

Estimates for risk factor realisations and true values.



Real-world example: setup

- Australian death and population data
- **Periods** $t \in \{1997, \ldots, 2011\}$.
- Eight **age categories** 50–54 years,. . . , 80-84 years and 85+ for each gender.
- Ten non-idiosyncratic **risk factors** (death causes) $\Lambda_1, \ldots, \Lambda_{10}$.
- In this setting optimisation over 362 parameters is required.

Using the extended CreditRisk$^+$ setup with log-linear trends for death probabilities and linear trends for weights, we estimate the model via **matching of moments** and **MCMC** (random walk Metropolis–Hastings within Gibbs) with 20 000 steps.

Lee–Carter model vs. our annuity model

- Given the number of living people $m_{\mathrm{a,g}}(t)$ as well as annual deaths $n_{\mathrm{a,g}}(t)$, for age a, gender g and years $t \in \{1, \ldots, T\}$, the **Lee–Carter model** models death rates $\hat{q}_{\mathrm{a,g}}(t) := n_{\mathrm{a,g}}(t)/m_{\mathrm{a,g}}(t)$ as

$$\log \hat{q}_{\mathrm{a,g}}(t) = a_{\mathrm{a,g}} + \kappa_t \, b_{\mathrm{a,g}} + \varepsilon_{\mathrm{a,g},t} \,, \quad \varepsilon_{\mathrm{a,g},t} \sim N(0, \sigma_\epsilon^2),$$

  with independent normal error terms $\varepsilon_{\mathrm{a,g},t}$.
- Using a suitable normalisations, e.g.

$$\sum \kappa_t = 0, \sum \beta_{\mathrm{a,g}} = 1$$

  $\hat{a}_{\mathrm{a,g}}$, $\hat{b}_{\mathrm{a,g}}$ and $(\hat{\kappa}_t)_{t \in \{1,\ldots,T\}}$ are derived via method of moments and singular value decompositions

$$\hat{a}_{\mathrm{a,g}} = \frac{1}{T} \sum_t \log \hat{q}_{\mathrm{a,g}}(t)$$

  and $\hat{\beta}$ is 1st left and $\hat{\kappa}_t$ is 1st right singular vectors in SVD of matrix $\log \hat{q}_{\mathrm{a,g}}(t) - \hat{a}_{\mathrm{a,g}}$.
- Then process for κ_t is estimated

$$\kappa_t = \kappa_{t-1} + \theta + \omega_t, \;\; \omega_t \sim N(0, \sigma_\omega^2)$$

Lee–Carter model: extensions

- **Multi-factor model extensions**, e.g.

$$\log \hat{q}_{\mathrm{a,g}}(t) = a_{\mathrm{a,g}} + \kappa_t^{(1)} b_{\mathrm{a,g}}^{(1)} + + \kappa_t^{(2)} b_{\mathrm{a,g}}^{(2)} + \varepsilon_{\mathrm{a,g},t}\,, \quad \varepsilon_{\mathrm{a,g},t} \sim N(0, \sigma_\epsilon^2),$$

- **Cohort effects**

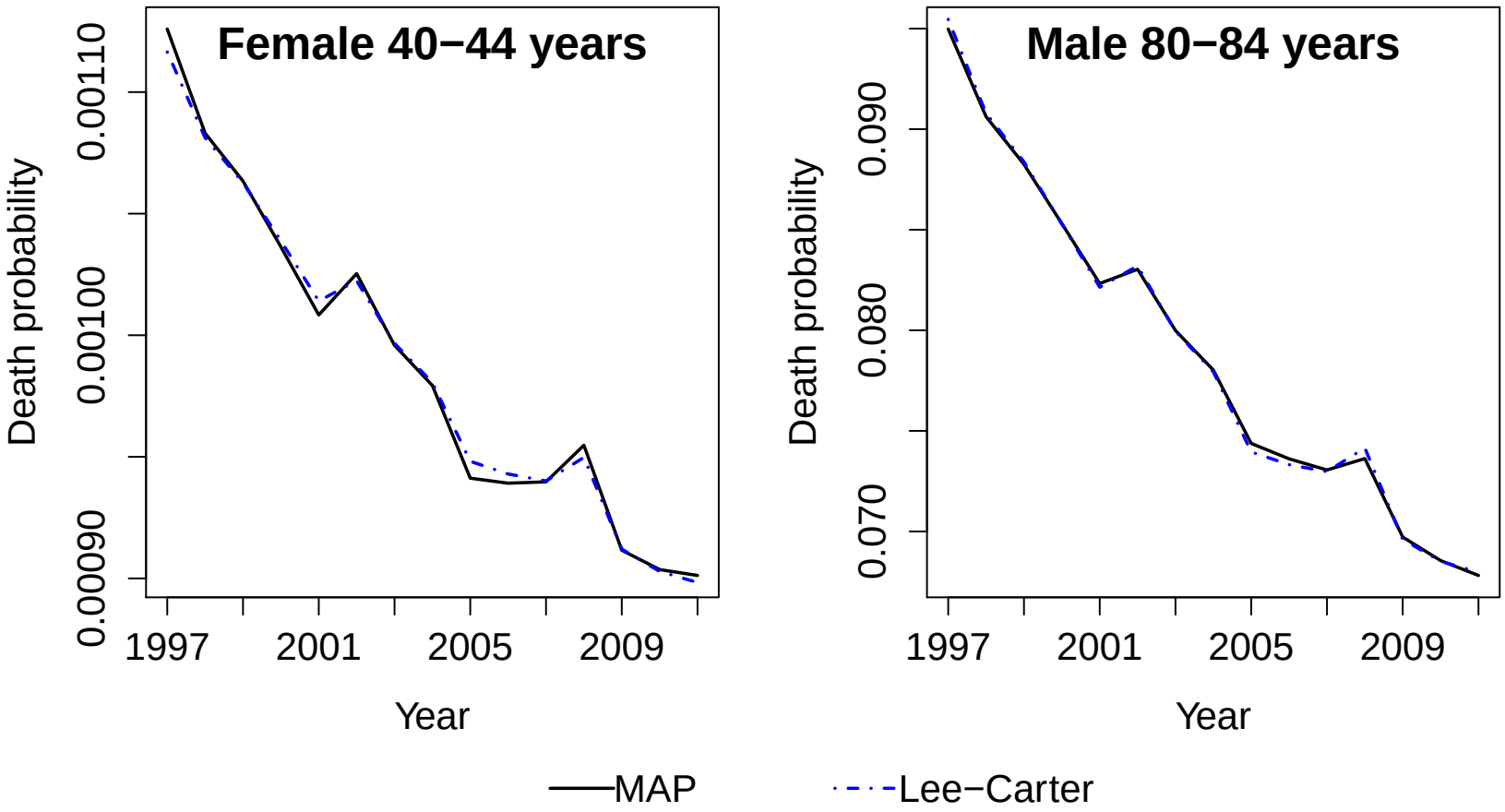
$$\log \hat{q}_{\mathrm{a,g}}(t) = a_{\mathrm{a,g}} + \kappa_t^{(1)} b_{\mathrm{a,g}}^{(1)} + + \kappa_{t-\mathrm{a}}^{(2)} b_{\mathrm{a,g}}^{(2)} + \varepsilon_{\mathrm{a,g},t}\,, \quad \varepsilon_{\mathrm{a,g},t} \sim N(0, \sigma_\epsilon^2),$$

- **State-space estimation**: Lee-Carter model is Gaussian state-space model and can be estimated using Kalman filter; Gibbs sampler can be derived for Lee Carter models and SMC can be developed for non-Gaussian extensions such as stochastic volatility, Fung, Peters and Shevchenko (2015).

Lee–Carter model vs. our annuity model

Consider our annuity model with one common risk factor $\Lambda_1(t)$ and weights $w_{\mathrm{a,g},1}(t) = 1$, for all $t \in \{1, \ldots, T\}$. Then, we expect

$$q^{\mathrm{LC}}_{\mathrm{a,g}}(t) \approx q^{\mathrm{MAP}}_{\mathrm{a,g}}(t)\lambda^{\mathrm{MAP}}_1(t), \quad t \in \{1, \ldots, T\}.$$



Estimation results: Weights and changes

Estimated weights and corresponding *linear* trends for different death causes using MCMC posterior mean estimates for males 50–54 years (left) and 80–84 years (right).

	Weight	Shift	Weight	Shift
Infectious & parasitic	2.86%	3.56%	1.38%	0.85%
Neoplasms	36.09%	-0.66%	32.23%	1.45%
Endocrine & nutritional	3.35%	-0.73%	4.64%	2.65%
Mental and behavioural	1.49%	1.72%	4.29%	4.79%
Nervous system	2.83%	1.43%	4.65%	2.66%
Circulatory	23.65%	-1.60%	32.13%	-3.15%
Respiratory system	3.55%	1.18%	10.61%	-0.27%
Digestive system	7.25%	1.90%	2.89%	0.65%
Injury and poisoning	15.84%	1.32%	2.95%	2.70%
Genitourinary system	0.75%	2.38%	2.71%	-0.74%

Estimation results: Weights forecast (nonlinear trends)

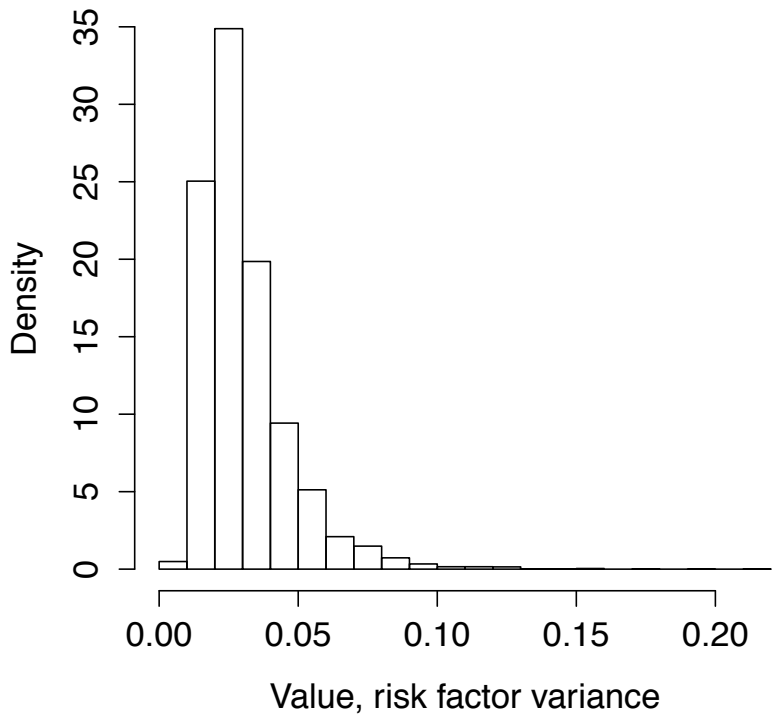
	male				female			
	60 to 64 years		80 to 84 years		60 to 64 years		80 to 84 years	
	2011	2031 (quant.)	2011	2031 (quant.)	2011	2031 (quant.)	2011	2031 (quant.)
neop.	0.499	0.547 $\binom{0.561}{0.531}$	0.324	0.378 $\binom{0.392}{0.364}$	0.592	0.648 $\binom{0.662}{0.629}$	0.263	0.303 $\binom{0.319}{0.288}$
circ.	0.228	0.116 $\binom{0.123}{0.109}$	0.325	0.173 $\binom{0.181}{0.164}$	0.140	0.060 $\binom{0.065}{0.055}$	0.342	0.149 $\binom{0.158}{0.140}$
ext.	0.056	0.062 $\binom{0.073}{0.053}$	0.026	0.028 $\binom{0.033}{0.024}$	0.072	0.069 $\binom{0.078}{0.060}$	0.100	0.126 $\binom{0.139}{0.113}$
resp.	0.051	0.036 $\binom{0.040}{0.032}$	0.106	0.092 $\binom{0.101}{0.083}$	0.038	0.037 $\binom{0.043}{0.032}$	0.051	0.068 $\binom{0.074}{0.061}$
endo.	0.044	0.062 $\binom{0.070}{0.055}$	0.047	0.077 $\binom{0.084}{0.070}$	0.036	0.051 $\binom{0.060}{0.043}$	0.054	0.080 $\binom{0.089}{0.071}$
dig.	0.041	0.036 $\binom{0.040}{0.031}$	0.027	0.020 $\binom{0.023}{0.018}$	0.035	0.032 $\binom{0.038}{0.026}$	0.024	0.023 $\binom{0.027}{0.020}$
nerv.	0.029	0.052 $\binom{0.061}{0.045}$	0.045	0.061 $\binom{0.068}{0.055}$	0.031	0.024 $\binom{0.029}{0.020}$	0.034	0.023 $\binom{0.027}{0.020}$
idio.	0.018	0.028 $\binom{0.034}{0.023}$	0.015	0.018 $\binom{0.020}{0.016}$	0.022	0.023 $\binom{0.028}{0.019}$	0.023	0.024 $\binom{0.027}{0.022}$
inf.	0.014	0.025 $\binom{0.033}{0.020}$	0.015	0.022 $\binom{0.027}{0.019}$	0.014	0.020 $\binom{0.027}{0.015}$	0.017	0.024 $\binom{0.028}{0.020}$
ment.	0.013	0.027 $\binom{0.036}{0.019}$	0.041	0.105 $\binom{0.130}{0.078}$	0.012	0.032 $\binom{0.046}{0.021}$	0.062	0.155 $\binom{0.188}{0.118}$
geni.	0.008	0.008 $\binom{0.010}{0.006}$	0.028	0.025 $\binom{0.028}{0.023}$	0.009	0.005 $\binom{0.006}{0.004}$	0.029	0.026 $\binom{0.028}{0.023}$

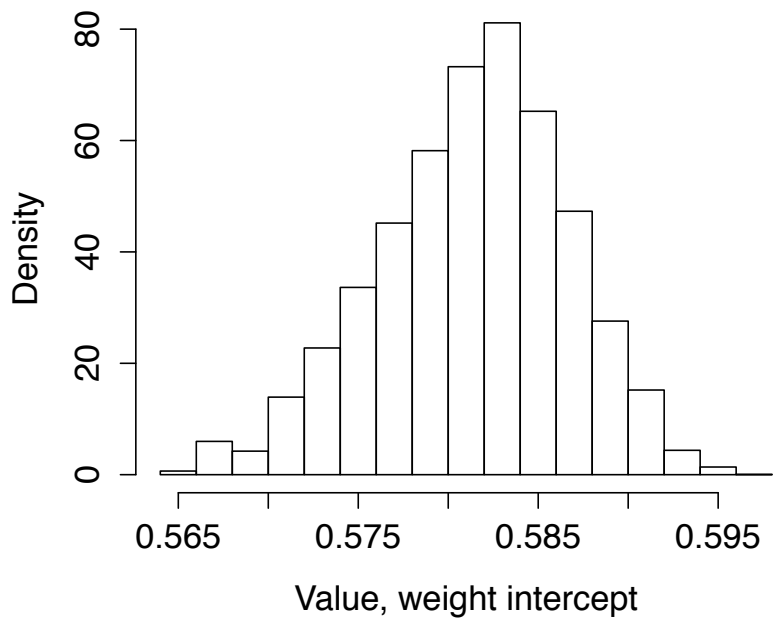
Leading death causes for years 2011 and 2051

		male		female	
		2011	2051	2011	2051
55–59 years	1.	neoplasms (0.469)	neoplasms (0.474)	neoplasms (0.603)	neoplasms (0.581)
	2.	circulatory (0.222)	infectious (0.092)	circulatory (0.112)	nervous (0.077)
	3.	external (0.085)	external (0.083)	respiratory (0.058)	not elsewhere (0.068)
65–69 years	1.	neoplasms (0.505)	neoplasms (0.575)	neoplasms (0.551)	neoplasms (0.609)
	2.	circulatory (0.226)	endocrine (0.082)	circulatory (0.162)	mental (0.112)
	3.	respiratory (0.072)	mental (0.075)	respiratory (0.083)	nervous (0.065)
75–79 years	1.	neoplasms (0.405)	neoplasms (0.466)	neoplasms (0.365)	neoplasms (0.378)
	2.	circulatory (0.277)	mental (0.185)	circulatory (0.271)	mental (0.245)
	3.	respiratory (0.100)	endocrine (0.098)	respiratory (0.103)	respiratory (0.108)
85+ years	1.	circulatory (0.395)	mental (0.329)	circulatory (0.441)	mental (0.503)
	2.	neoplasms (0.217)	neoplasms (0.216)	neoplasms (0.131)	circulatory (0.092)
	3.	respiratory (0.115)	circulatory (0.133)	mental (0.101)	neoplasms (0.090)

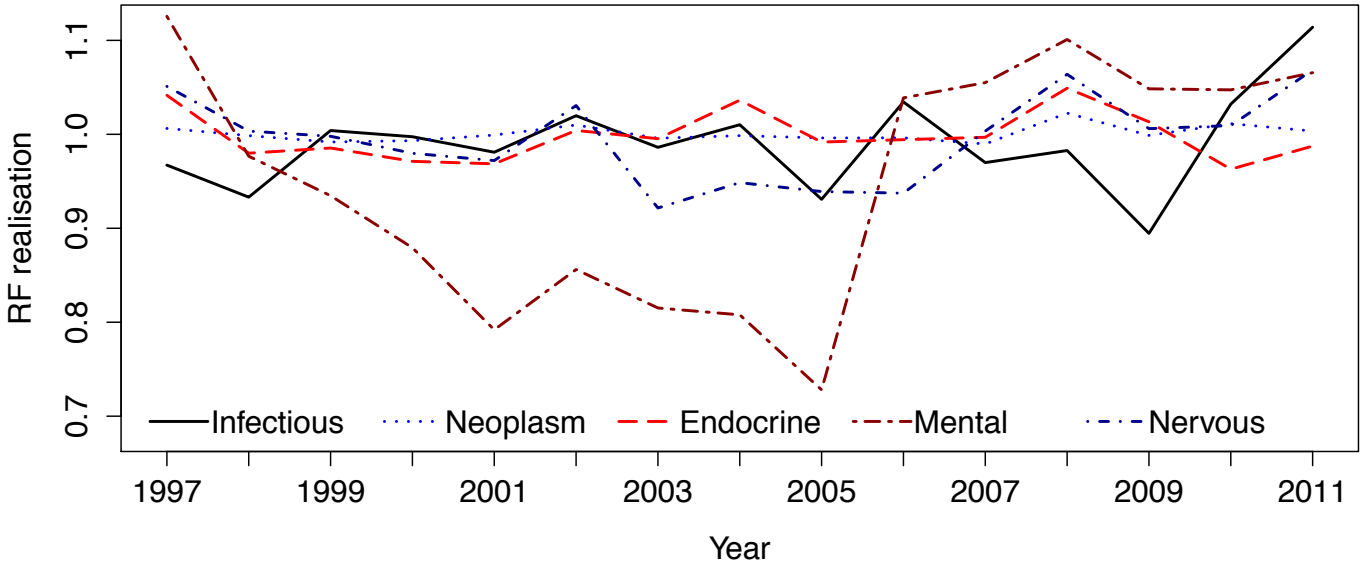
Estimation results: MCMC density histograms

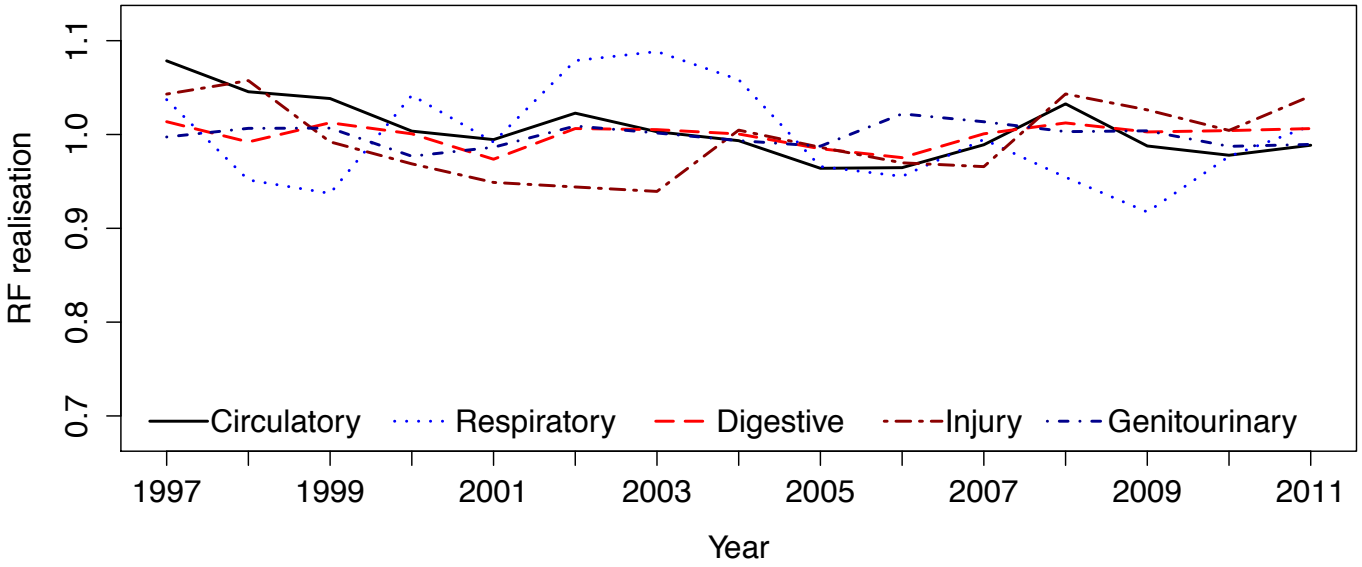
Density histograms of MCMC chains for the variance of risk factor for mental and behavioural disorders as well as for weight intercept for females aged 55 to 59 years of death cause neoplasms (right).





Estimation results: Risk Factors





Forecast: death probabilities and death causes weights

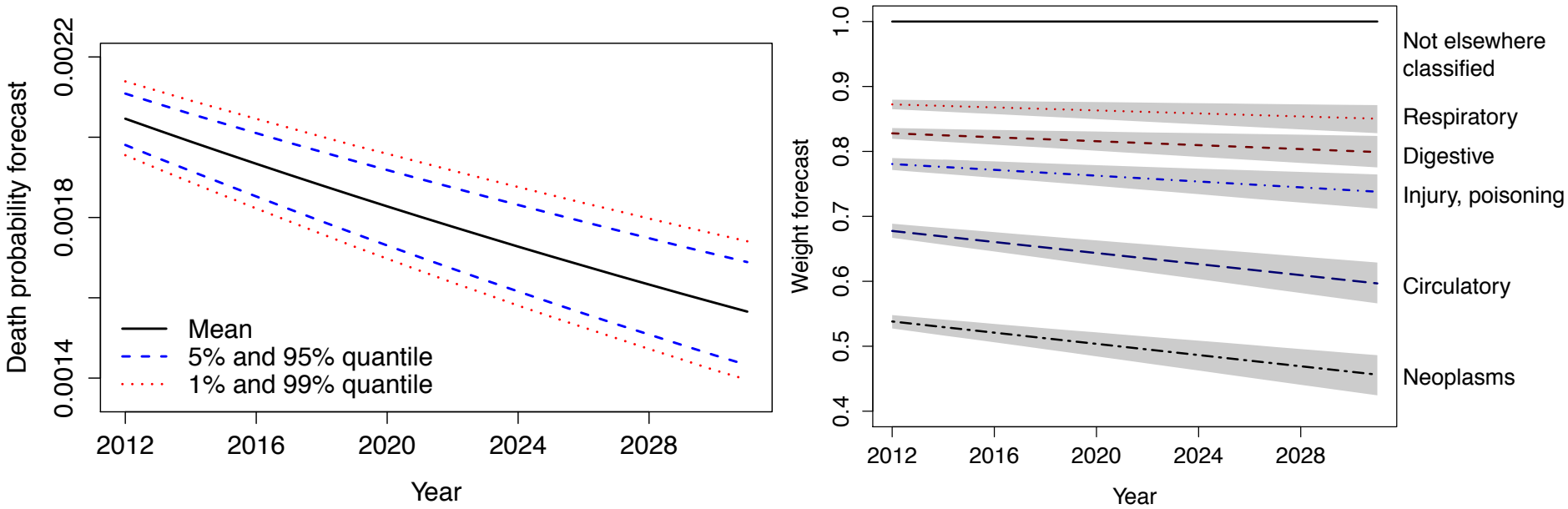


Figure : Forecasted death probabilities and cumulative weights for various death causes for females aged 50-54; shaded area correspond to 5% and 95% quantiles

Forecast for death probabilities

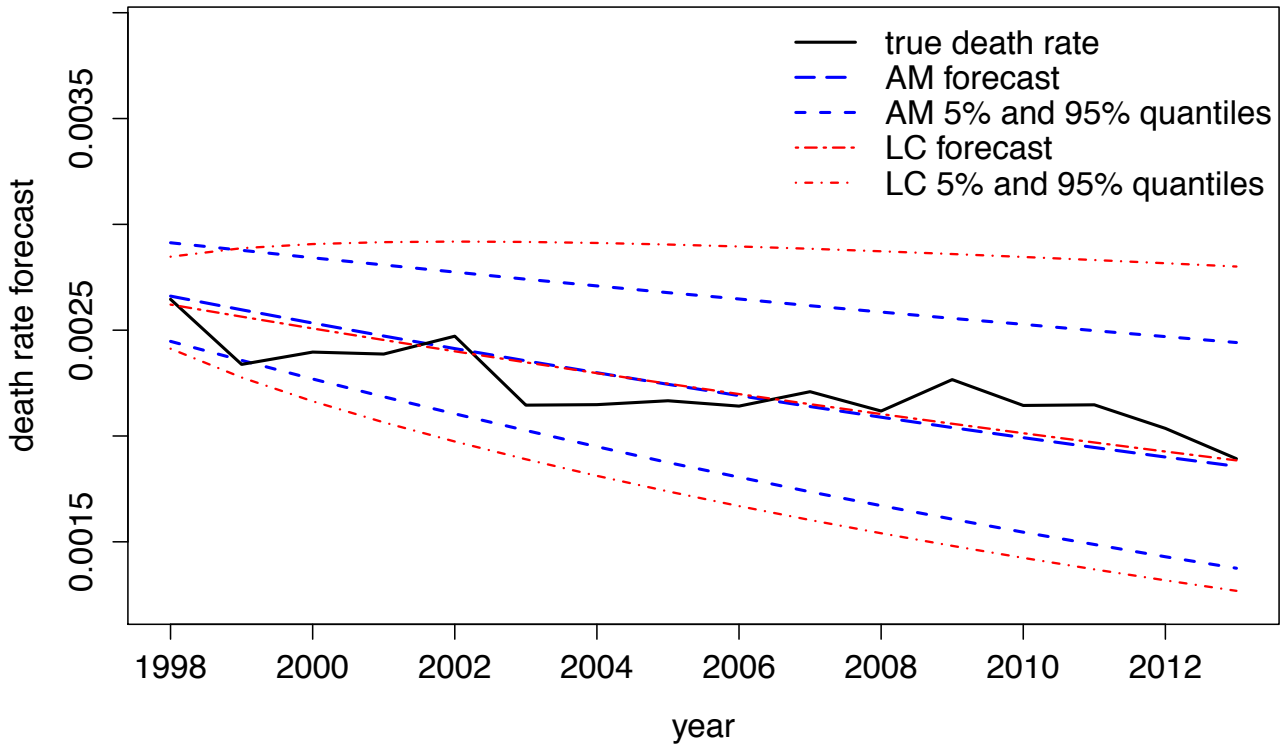
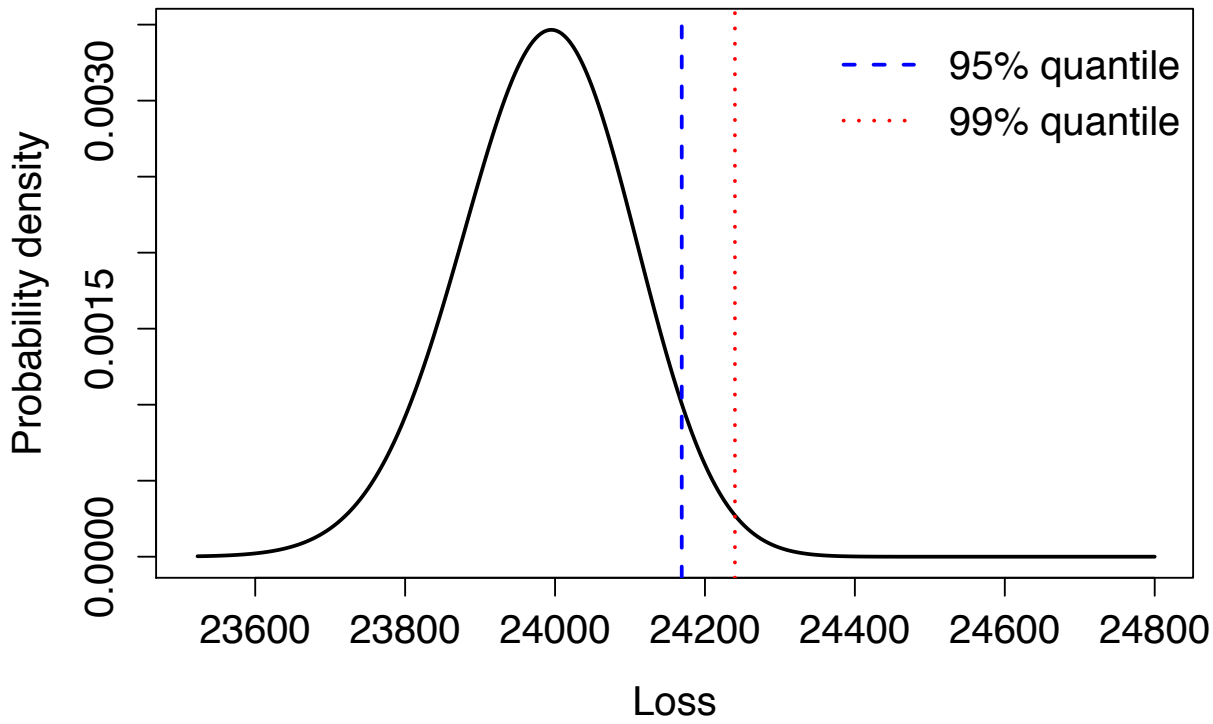


Figure : Forecasted death probabilities and 90% confidence intervals using Australian death and population data for the years 1963 to 1997.

Parameter uncertainty: portfolio loss distribution

- Australian data with the same setup as before.
- Let each age category and gender have 100 policyholders with annual deterministic annuity payments $X_i = 11, \ldots, 20$.
- Derive loss distribution $L = \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} \sum_{j=1}^{N_i(T+1)} X_{i,j}$ with extended CreditRisk$^+$ where $X_{i,j} \sim X_i$.



Parameter uncertainty: Distribution of quantiles

As we are using MCMC, we can derive (approximatively) distributions of quantiles of L, i.e., we can quantify parameter risk.

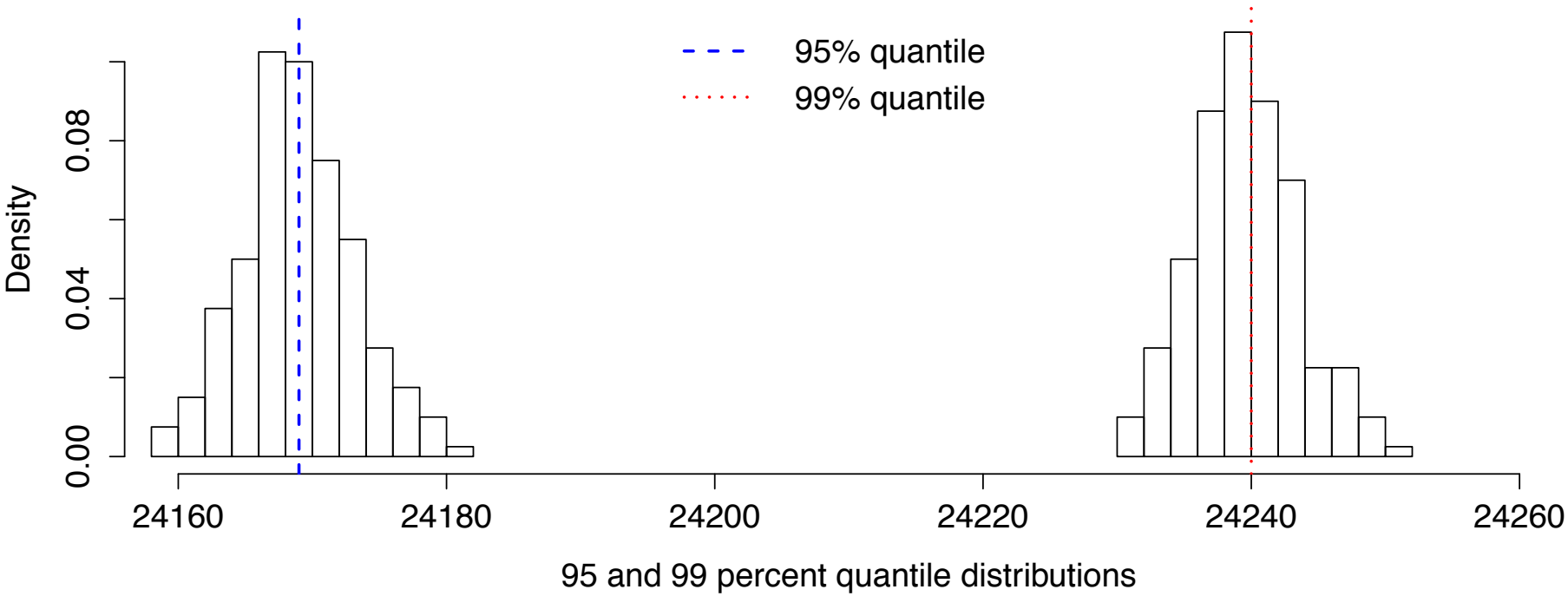
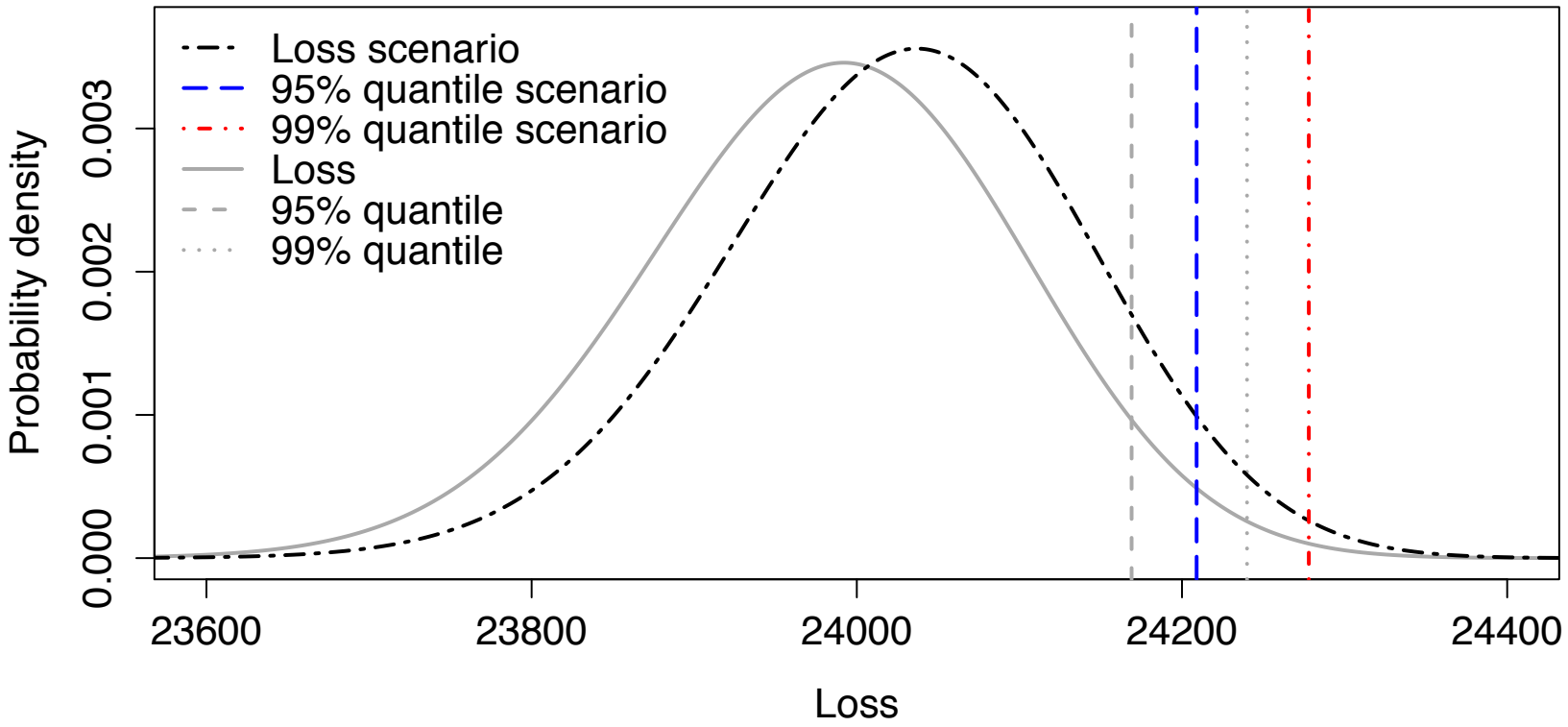


Figure : Distributions of 95 and 99 percent quantile based on MCMC chain realisations.

Scenario analysis

If we assume that deaths due to cancer decrease by 25% over all age categories next year due to better medication, then we get the following shifted distribution of L.



Model validation

- validation via cross-covariance $\mathrm{Cov}(N_{\mathrm{a,g},k}, N_{\mathrm{a}',\mathrm{g}',k})$
- validation via independence of death counts for different death causes $\mathrm{Cov}(N_{\mathrm{a,g},k}, N_{\mathrm{a}',\mathrm{g}',k'}) = 0$
- validation via serial correlation, $N_{\mathrm{a,g},k}(t), t = 1, \ldots, T$ are independent
- validation via risk factor realisations (Λ_k are from Gamma with mean=0 and variance β_k).
- model selection (reduction of factors): AIC, BIC, DIC

Model Forecast: Male, Australia

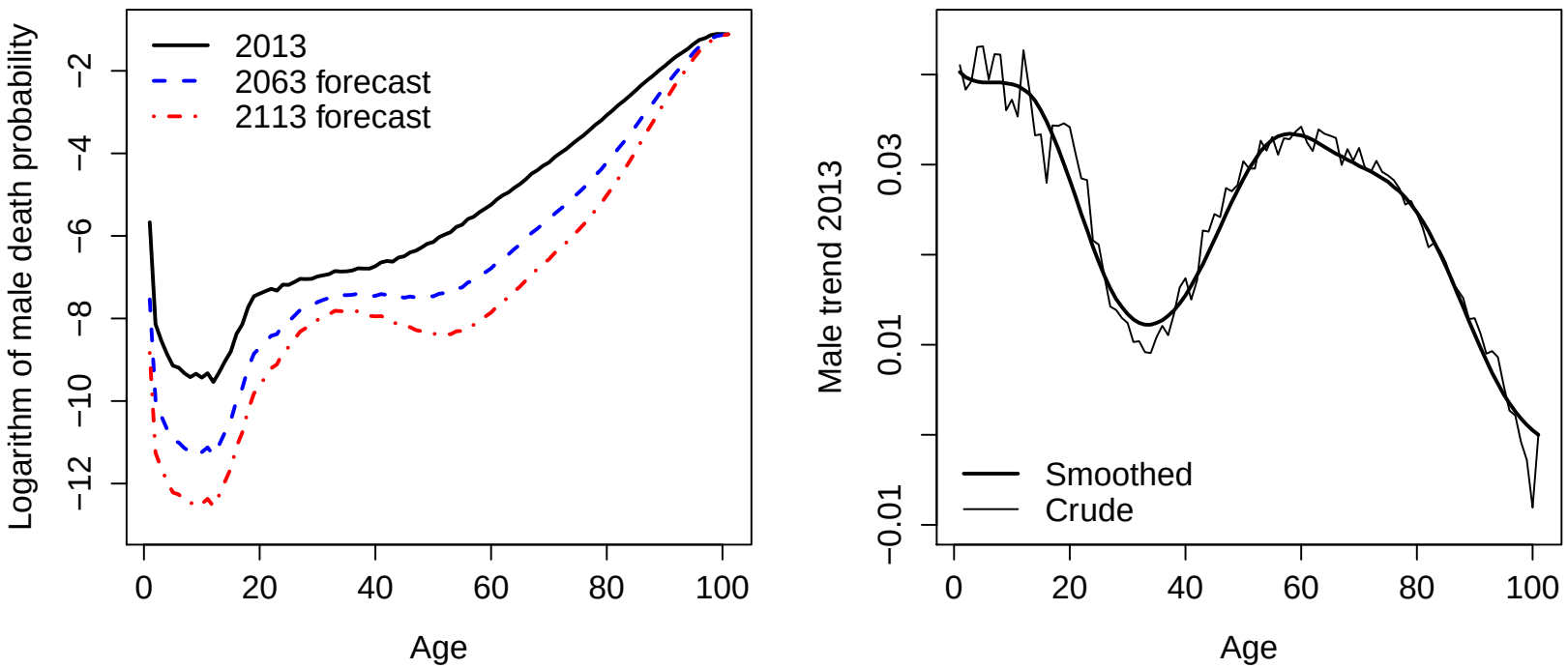


Figure : Australian logarithmic death probabilities as well as forecasts, i.e., ($q_{a,g}(2013)$; $q_{a,g}(2063)$; $q_{a,g}(2113)$, based on our annuity model using Australian from 1971 to 2013 (left) as well as correspoding smoothed mortality trends $b_{a,g}$ (right).

Model Forecast: Female, Australia

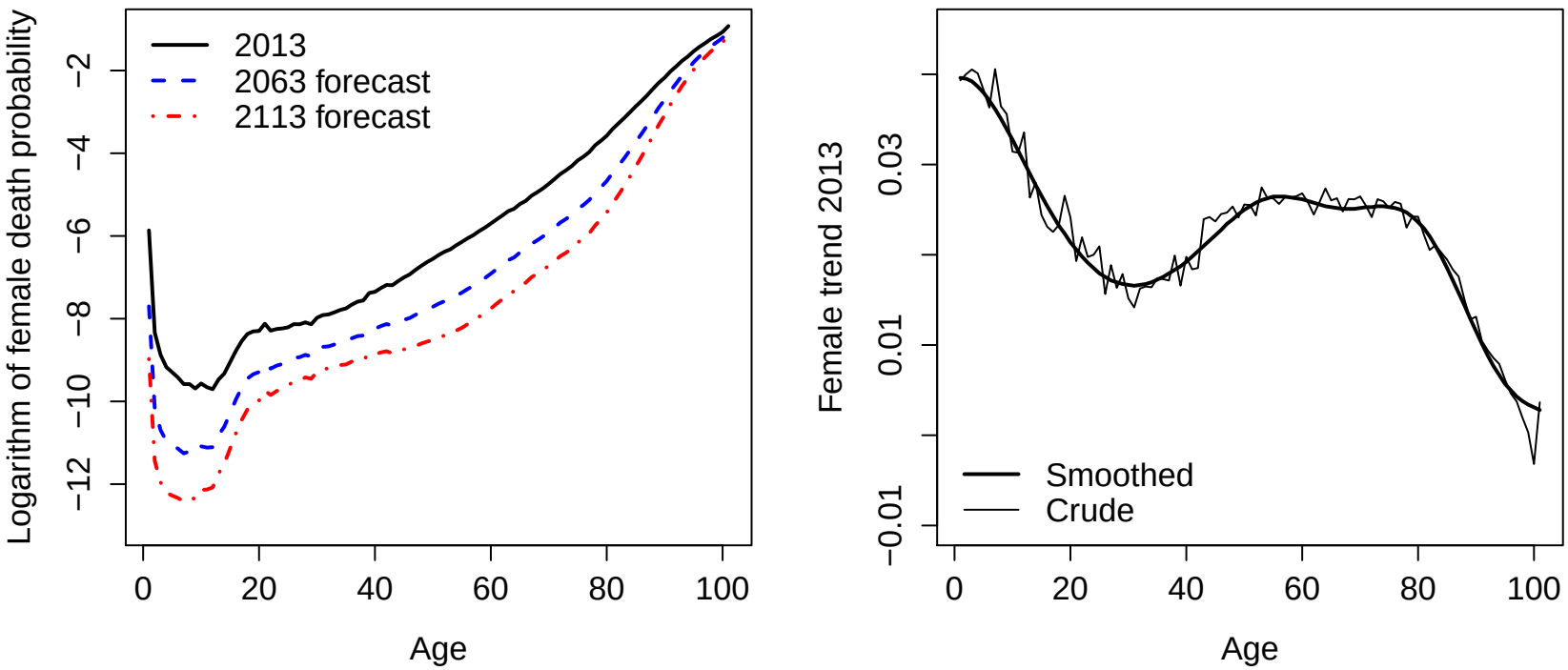
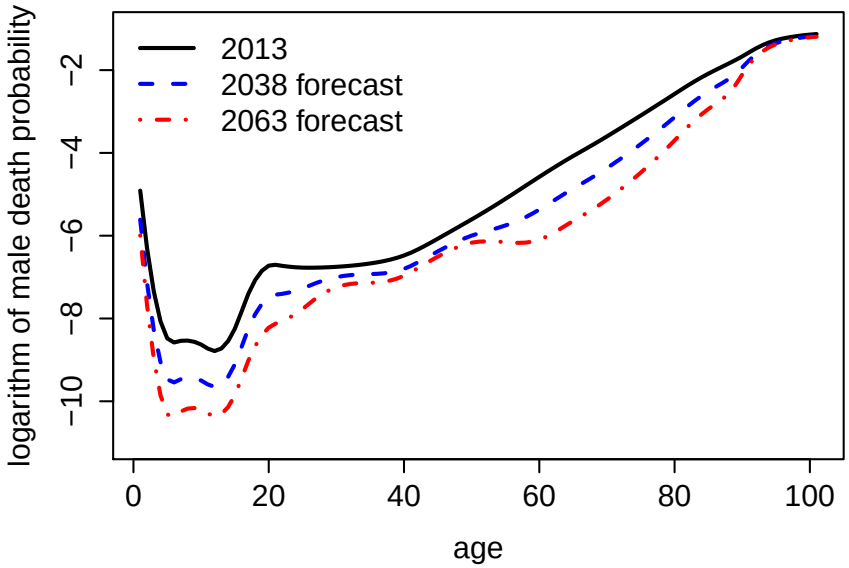
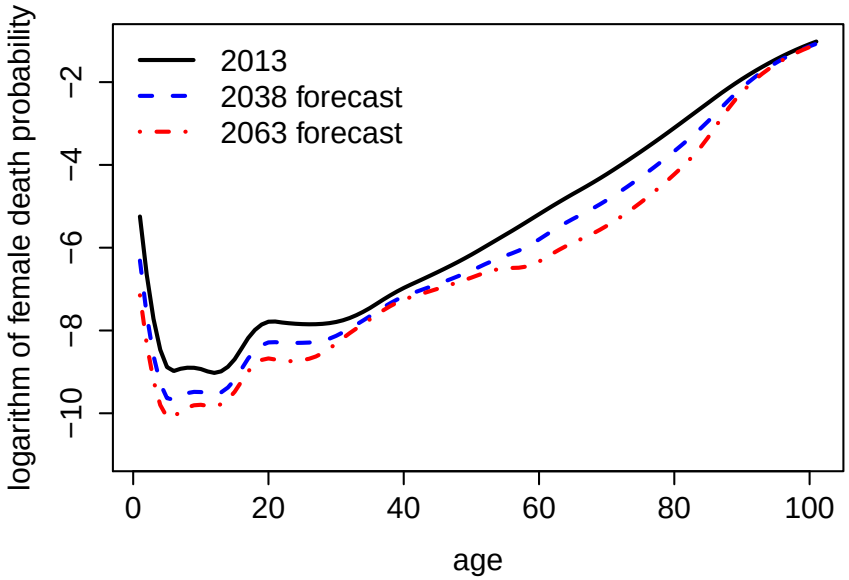
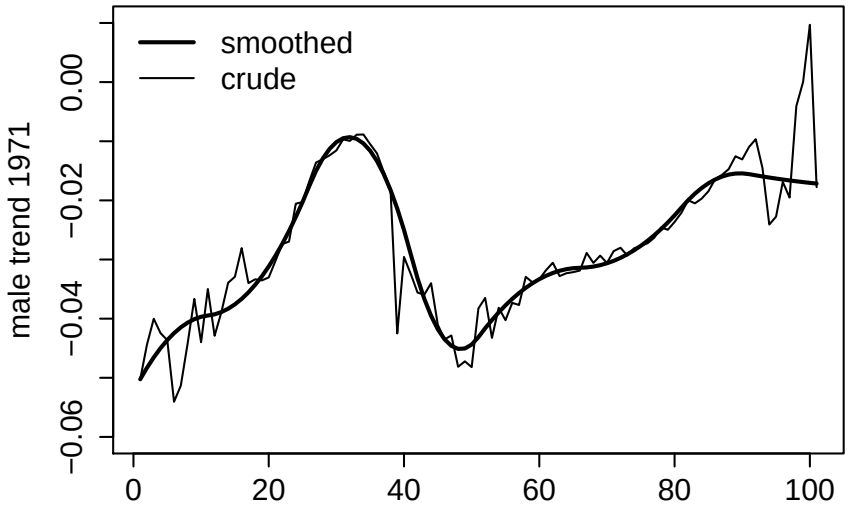


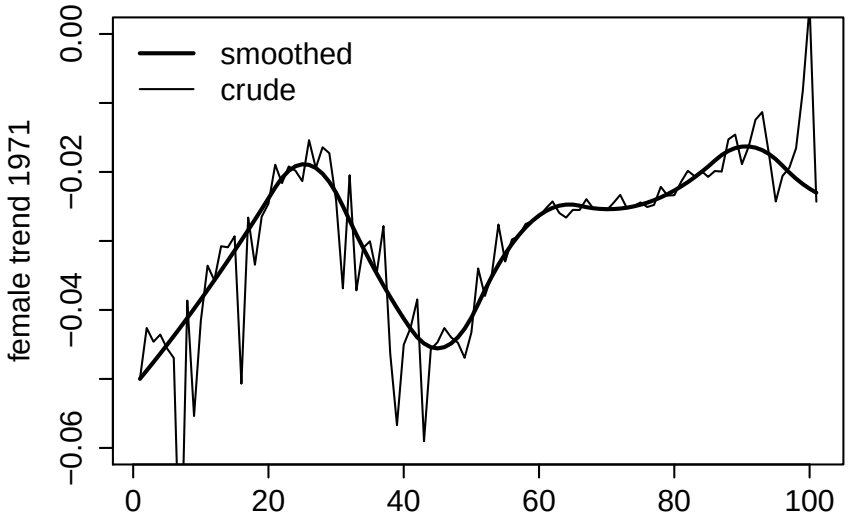
Figure : Australian logarithmic death probabilities as well as forecasts, i.e., $q_{a,g}(2013)$; $q_{a,g}(2063)$; $q_{a,g}(2113)$, based on our annuity model using Australian from 1971 to 2013 (left) as well as corresponding smoothed mortality trends $b_{a,g}$ (right).

Nonlinear Trends and Forecast for death probabilities, Australia









Model Forecast: Male, Australia

TABLE D.1. 2013 Australian male life table.

Age x	Annual death probability $q_x(2013)$	Survivors up to x $l_x(2013)$	Deaths from x to $x+1$ $d_x(2013)$	Mortality trend $1-\exp(-b_x)$	EFLT with trend $e_x(2013)$	EFLT no trend $e^*_x(2013)$
0	0.003448	100000	345	0.0411	89.56	79.81
1	0.000293	99655	29	0.0405	88.79	79.09
2	0.000196	99626	20	0.0402	87.74	78.11
3	0.000141	99606	14	0.0400	86.68	77.13
4	0.000107	99592	11	0.0399	85.61	76.14
.....						
70	0.016734	83785	1402	0.0300	15.92	14.69
71	0.018522	82383	1526	0.0297	15.05	13.94
72	0.020361	80857	1646	0.0293	14.21	13.21
73	0.022931	79211	1816	0.0289	13.38	12.48
74	0.025593	77394	1981	0.0285	12.58	11.77
75	0.028412	75413	2143	0.0279	11.80	11.08

Model Forecast: Female, Australia

TABLE D.2. 2013 Australian female life table.

Age x	Annual death probability $q_x(2013)$	Survivors up to x $l_x(2013)$	Deaths from x to $x+1$ $d_x(2013)$	Mortality trend $1-\exp(-b_x)$	EFLT with trend $e_x(2013)$	EFLT no trend $e^*_x(2013)$
0	0.002837	100000	284	0.0404	91.91	84.13
1	0.000240	99716	24	0.0403	91.11	83.37
2	0.000140	99692	14	0.0400	90.07	82.39
3	0.000104	99678	10	0.0394	89.02	81.41
4	0.000092	99668	9	0.0387	87.97	80.41
.....						
70	0.009823	90059	885	0.0256	18.53	17.33
71	0.011078	89174	988	0.0256	17.61	16.50
72	0.012152	88186	1072	0.0257	16.70	15.68
73	0.013485	87115	1175	0.0257	15.80	14.88
74	0.015398	85940	1323	0.0256	14.92	14.08
75	0.016883	84617	1429	0.0255	14.05	13.30

Further applications

- **Population forecasts**.
- Effects of **scenarios** where death rates of certain death causes suddenly spike by $x \cdot 100$ percent within one year can be derived.
- Our model can be generalised to individual losses $Y_{i,k}$ depending on death cause k which allows **modelling of new life-insurance contracts**.
- Certain **dependence structures for risk factors** can be assumed while still being able to calculate loss distributions exactly via iterated Panjer's recursion.

References I

Creditrisk$^+$: A credit risk management framework. Technical report, Credit Suisse First Boston, 1997.

M. Cruz, G.W. Peters, and P.V. Shevchenko. *Fundamental Aspects of Operational Risk and Insurance Analytics: a Handbook of Operational Risk*. Wiley, 2015a.

M. Cruz, G.W. Peters, and P.V. Shevchenko. *Advances in Heavy Tailed Risk Modeling: a Handbook of Operational Risk*. Wiley, 2015b.

S. Fung, G.W. Peters, and P.V. Shevchenko. A bayesian approach to mortality modeling and implications for annuity pricing. *Preprint, to appear in MODSIM 2015 conference proceedings*, 2015.

S. Gerhold, U. Schmock, and R. Warnung. A generalization of panjer's recursion and numerically stable risk aggregation. *Finance and Stochastics*, 14(1):81–128, 2010.

M. Gundlach and F. Lehrbass, editors. *CreditRisk$^+$ in the Banking Industry*. Springer Finance. Springer-Verlag, Berlin, 2004. ISBN 3-540-20738-4.

J. Hirz, U. Schmock, and P. V. Shevchenko. Modelling annuity portfolios and longevity risk with extended creditrisk+. *Preprint, arxiv: 1505.04757 available on http://arxiv.org/abs/1505.04757*, 2015.

References II

R. Kainhofer, M. Predota, and U. Schmock. The new Austrian annuity valuation table AVÖ 2005R. *Mitteilungen der Aktuarvereinigung sterreichs*, 13:55–135, 2006.

U. Schmock. Modelling Dependent Credit Risks with Extensions of CreditRisk$^+$ and Application to Operational Risk (Lecture Notes), 2014. URL `http://www.fam.tuwien.ac.at/~schmock/notes/ExtensionsCreditRiskPlus.pdf`.

P. V. Shevchenko, J. Hirz, and U. Schmock. Mortality modelling using extended creditriskplus. *Preprint, to appear in MODSIM 2015 conference proceedings*, 2015.

P.V. Shevchenko. *Modelling operational risk using Bayesian inference*. Springer-Verlag, 2011.

P. Vellaisamy and B. Chaudhuri. Poisson and compound Poisson approximations for random sums of random variables. *J. Appl. Probab.*, 33(1):127–137, 1996. ISSN 0021-9002.

Thank you for your attention!

Prof Pavel Shevchenko

Senior Principal Research Scientist, CSIRO Australia
Adjunct Professor at UNSW and UTS
Email: Pavel.Shevchenko@csiro.au
http://people.csiro.au/S/P/Pavel-Shevchenko.aspx