Symmetric α-Stable Random Variables, Fractional Calculus, and the Fourier-Mellin Triangle

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This Talk in a Nutshell

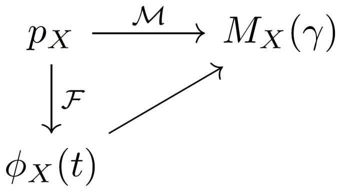
- This talk is about the interaction between:
  1. the probability density function of symmetric α-stable ($S\alpha S$) random variables;
  2. the Fourier transform;
  3. and the Mellin transform.

This Talk in a Nutshell

- This interaction is mediated by techniques from fractional calculus.
- Many of these ideas have been developed by Di Paolo et al in
  1. Cottone and Di Paolo (2009), *On the use of fractional calculus for the probabilistic characterization of random variables*.
  2. Di Paolo and Pinnola (2012), *Riesz fractional integrals and complex fractional moments for the probabilistic characterization of random variables*.

This Talk in a Nutshell

- The following diagram—the Fourier-Mellin triangle—plays a key role:



Why α-Stable Random Variables?

- Distributions with heavy tails are not well-modeled as Gaussian.
- α-stable distributions are a relatively tractable alternative.
    - $\mathbb{P}(X > \lambda) \sim C\lambda^{-\alpha}$
- Applications:
    1. Interference and noise modeling (e.g., in wireless radio communications).
    2. Asset returns in finance.

Symmetric α-Stable Random Variables

- An important sub-class are the symmetric α-stable ($S\alpha S$) random variables.
- One way $S\alpha S$ random variables arise is the LePage series

$$X = \sum_{i=1}^{\infty} r_i^{-\alpha} g_i,$$

where

1. $\{r_i\}$ are the arrival times of the Poisson process with rate 1;
2. $\{g_i\}$ are symmetric random variables $X \stackrel{d}{=} -X$, independent of $\{r_i\}$;
3. $\mathbb{E}[g_i^{\alpha}] < \infty$.

- In applications this type of sum is called *shot noise*.

Symmetric α-Stable Random Variables

- A common definition of $S\alpha S$ random variables is via

$$AX_1 + BX_2 \stackrel{d}{=} CX, \quad X \stackrel{d}{=} -X,$$

where

1. X_1, X_2 are independent copies of X;
2. $A^\alpha + B^\alpha = C^\alpha$, for some $\alpha \in (0, 2]$.

Symmetric α-Stable Random Variables

- $S\alpha S$ random variables are also infinitely divisible.
- For $n \geq 2$, there is a $C_n > 0$ such that

$$X_1 + \cdots X_n \overset{d}{=} C_n X, \quad X \overset{d}{=} -X,$$

where

1. $X_1, \ldots, X_n$ are independent copies of X;
2. $C_n = n^{1/\alpha}$.

Examples of Symmetric α-Stable Random Variables

1. Gaussian distribution ($X \sim S_2(\sigma, 0, 0)$):

$$p_X(x) = \frac{1}{\sqrt{4\pi\sigma^2}} e^{-\frac{x^2}{4\sigma^2}}$$
$$\mathbb{E}[e^{itX}] = e^{-\sigma^2 t^2}$$

2. Cauchy distribution ($X \sim S_1(\sigma, 0, 0)$):

$$p_X(x) = \frac{1}{\pi\sigma}\left[\frac{\sigma^2}{x^2 + \sigma^2}\right]$$
$$\mathbb{E}[e^{itX}] = e^{-\sigma|t|}$$

In general, $S\alpha S$ random variables do not have closed form densities.

The Characteristic Function of $S\alpha S$ Random Variables

- While the density of $S\alpha S$ random variables is difficult to work with, the characteristic function

$$\mathbb{E}[e^{itX}] = \int_{-\infty}^{\infty} p_X(x)e^{itX}\,dx$$

  is known in closed form.

- In particular, let $X \sim S_\alpha(\sigma, 0, 0)$. Then,

$$\phi_X(t) = \mathbb{E}[e^{itX}] = e^{-\sigma^\alpha |t|^\alpha}.$$

- This result is very useful.

Basic Properties of $S\alpha S$ Random Variables

1. Let X_1, X_2 be independent with $X_i \sim S_\alpha(\sigma_i, 0, 0)$. Then,

$$X_1 + X_2 \sim S_\alpha\left((\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}, 0, 0\right).$$

2. Let $X \sim S_\alpha(\sigma, 0, 0)$ and $a \in \mathbb{R}$. Then,

$$X + a \sim S_\alpha(\sigma, 0, a).$$

3. Let $X \sim S_\alpha(\sigma, 0, 0)$ and $a \in \mathbb{R} \setminus \{0\}$. Then,

$$aX \sim S_\alpha(|a|\sigma, 0, 0).$$

Basic Properties of $S\alpha S$ Random Variables

1. Let $X \sim S_\alpha(\sigma, 0, 0)$. Then,

$$\mathbb{P}(X > \lambda) \sim \sigma^\alpha \frac{C_\alpha}{2} \lambda^{-\alpha}.$$

2. Let $X \sim S_\alpha(\sigma, 0, 0)$ and $0 < p < \alpha$. Then,

$$\mathbb{E}[|X|^p] = \frac{2^{p+1}\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(-\frac{p}{\alpha}\right)}{\alpha\sqrt{\pi}\Gamma\left(-\frac{p}{2}\right)}\sigma^p.$$

The Mellin Transform

- Instead of taking the Fourier transform (i.e., the characteristic function), we can consider the Mellin transform.
- For α-stable distributions, this was first done by Zolotarev in 1957.
- Since $S\alpha S$ densities are absolutely continuous functions, the Mellin transform is

$$\mathcal{M}[p_X(x)](\gamma) = \int_0^\infty p_X(x)x^{\gamma-1}dx, \ \gamma \in \mathbb{C}.$$

(More generally, the Mellin-Stieltjes transform is required.)

The Mellin Transform

- The Mellin transform can be related to the Fourier transform through a change of variables.
- Consider the operator $T : f(x) \to f(e^x), f \in L^1$; i.e.,

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

- Define the T-norm

$$\|f\|_{\mathcal{M}_c} = \int_0^{\infty} |f(x)| x^{c-1} | dx, \tag{1}$$

where c is chosen to ensure convergence for the class of functions f we are interested in.

The Mellin Transform

- L^1 functions with finite T-norm

$$\int_0^{\infty} |f(x)|x^{c-1}|dx < \infty \tag{2}$$

form a function space.

- On this space, the Fourier transform

$$\mathcal{F}(Tf)(t) = \int_{-\infty}^{\infty} f(e^x)e^{itx}dx.$$

is well defined. (More on this in Butzer and Jansche (1997)).

The Mellin Transform

- Now consider

$$\mathcal{F}[Tf]\left(\frac{\eta - c}{i}\right) = \int_{-\infty}^{\infty} f(e^x)e^{(\eta - c)x}dx, \;\; c \geq 0.$$

- Let $y = e^x$, which gives

$$\begin{aligned}\mathcal{F}(Tf)\left(\frac{\eta - c}{i}\right) &= \int_0^{\infty} f(y)e^{(\eta - c)\log y}\frac{1}{y}dy \\ &= \int_0^{\infty} f(y)y^{-c}y^{\eta}\frac{1}{y}dy \\ &= \int_0^{\infty} f^*(y)y^{\eta - 1}dy, \;\; f^*(y) = f(y)y^{-c},\end{aligned}$$

which is the Mellin transform of $f^*(y)$.

The Mellin Transform

- This means that for functions with finite T-norm

$$\|f\|_{\mathcal{M}_c} = \int_0^\infty |f(x)|x^{c-1}dx, \tag{3}$$

  we can use theorems for the Fourier transform of L^1 functions.

- E.g., the Fourier inversion theorem (more on this later).

The Mellin Transform

- The Mellin transform has a nice property.
- Consider the product of two random variables $Z = XY$. Then,

$$\mathcal{M}[f_Z](s) = \mathcal{M}[f_X](s)\mathcal{M}[f_Y](s). \tag{4}$$

- This means that the Mellin transform plays a similar role for products of random variables as the Fourier transform plays for sums.

The Mellin Transform

- The Mellin transform has an intimate link to fractional calculus.
- To see this, we first overview some basic ideas in fractional calculus.

A Brief Overview of Fractional Calculus

- The starting point for fractional calculus is to generalize derivatives

$$\frac{d^n}{dx^n} f(x), \;\; n \in \mathbb{N}$$

to the case where $n \in \mathbb{R}$; e.g.,

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}} f(x).$$

- To see how this might work, consider $f(x) = x^p$.
- We have,

$$\frac{d^n}{dx^n}x^p = p(p-1)\cdots(p-n+1)x^{p-n}$$
$$= \frac{p!}{(p-n)!}x^{p-n} = \frac{\Gamma(p+1)}{\Gamma(p-n+1)}x^{p-n}.$$

- Using the properties of the Gamma function

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt,$$

it is possible to analytically continue to yield

$$\frac{d^q}{dx^q}x^p = \frac{\Gamma(p+1)}{\Gamma(p-q+1)}x^{p-q},$$

with $q \in \mathbb{R}$ (being careful with $q = -1, -2, \ldots$).

A Brief Overview of Fractional Calculus

- An important way of generalizing is via the Riemann-Liouville fractional integrals

$$(I_{\pm}^{\gamma} f)(x) = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} \zeta^{\gamma-1} f(x \mp \zeta) d\zeta.$$

- The Riemann-Liouville fractional derivatives are then

$$(D_{\pm}^{\gamma} f)(x) = \frac{(\pm 1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_0^{\infty} \zeta^{n-\gamma-1} f(x \mp \zeta) d\zeta,$$

  where $\gamma \in \mathbb{C}$ and $n = [\rho] + 1$, where $\rho = \mathrm{Re}(\gamma)$.
- This agrees with gamma function-based definition for the monomial example developed in the previous slide.

A Brief Overview of Fractional Calculus

- The Riemann-Liouville fractional integral satisfies the semigroup property:

$$I_+^{\alpha} I_+^{\beta} \psi = I_+^{\alpha+\beta} \psi, \quad I_-^{\alpha} I_-^{\beta} \psi = I_-^{\alpha+\beta} \psi, \tag{5}$$

where $\alpha, \beta > 0$.

The Link Between Fractional Calculus and the Mellin Transform

- A key observation is the link between the Riemann-Liouville fractional integral and the Mellin transform.
- Recall the Mellin transform is

$$\mathcal{M}[f(x)](\gamma) = \int_0^\infty p_X(x)x^{\gamma-1}dx,\ \gamma \in \mathbb{C}.$$

- The Riemann-Liouville fractional integral is

$$(I_\pm^\gamma f)(x) = \frac{1}{\Gamma(\gamma)}\int_0^\infty \zeta^{\gamma-1}f(x \mp \zeta)d\zeta.$$

- That is,

$$\mathcal{M}[f(x \mp \zeta)](\gamma) = \Gamma(\gamma)(I_\pm^\gamma f)(x).$$

A Key Identity

- For standard integrals, we have

$$\mathcal{F}\left[\int_{-\infty}^{x} f(\tau)d\tau\right](t) = \frac{\mathcal{F}[f](t)}{-it}$$

- This generalizes:

$$\mathcal{F}[I_+^\gamma f](t) = (-it)^{-\gamma}\mathcal{F}[f](t).$$

A Key Identity

- Take $0 < \rho < 1$. We need to take the Fourier transform of the fractional integral; i.e.,

$$\mathcal{F}[I_+^\gamma f](t) = \int_{-\infty}^{\infty} e^{itx} \frac{1}{\Gamma(\gamma)} \int_0^\infty \zeta^{\gamma-1} f(x-\zeta) d\zeta dx.$$

- Some basic manipulations yield

$$\mathcal{F}[I_+^\gamma f](t) = \frac{\mathcal{F}[f](t)}{\Gamma(\gamma)} \int_0^\infty e^{it\zeta} \zeta^{\gamma-1} d\zeta.$$

- A useful identity tells us that

$$\int_0^\infty e^{it\zeta} \zeta^{\gamma-1} = \Gamma(\gamma)(-it)^{-\gamma},$$

where taking the principle value we understand that

$$(-it)^{-\gamma} = \exp\left(-\gamma \log|t| + \frac{\gamma\pi i}{2}\mathrm{sgn}(t)\right).$$

A Key Identity

- Leading us to

$$\mathcal{F}[I_+^{\gamma} f](t) = (-it)^{-\gamma} \mathcal{F}[f](t).$$

- What does it mean?

Recovering the Mellin Transform

- The observation that

$$\mathcal{F}[I_+^\gamma f](t) = (-it)^{-\gamma}\mathcal{F}[f](t).$$

means that

$$\begin{aligned}(I_+^\gamma f)(t) &= \mathcal{F}^{-1}\left[(-it)^{-\gamma}\mathcal{F}[f](t)\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}(-it)^{-\gamma}\mathcal{F}[f](t)dt.\end{aligned}$$

Recovering the Mellin Transform

- The observation that

$$\mathcal{F}[I_+^{\gamma} f](t) = (-it)^{-\gamma}\mathcal{F}[f](t).$$

means that

$$\begin{aligned}(I_+^{\gamma} f)(t) &= \mathcal{F}^{-1}\left[(-it)^{-\gamma}\mathcal{F}[f](t)\right]\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}(-it)^{-\gamma}\mathcal{F}[f](t)dt.\end{aligned}$$

- As such,

$$(I_+^{\gamma} f)(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-it)^{-\gamma}\mathcal{F}[f](t)dt.$$

- Now recall that

$$\Gamma(\gamma)(I_+^{\gamma} f)(x) = \int_0^{\infty}\zeta^{\gamma-1}f(x-\zeta)d\zeta.$$

Recovering the Mellin Transform

- Putting it all together:

$$\int_0^\infty \zeta^{\gamma-1} f(-\zeta) d\zeta = \frac{\Gamma(\gamma)}{2\pi} \int_{-\infty}^\infty (-it)^{-\gamma} \mathcal{F}[f](t) dt.$$

- When f is even, we then have

$$\begin{aligned}\mathcal{M}[f](\gamma) &= \int_0^\infty \zeta^{\gamma-1} f(\zeta) d\zeta \\ &= \frac{\Gamma(\gamma)}{2\pi} \int_{-\infty}^\infty (-it)^{-\gamma} \mathcal{F}[f](t) dt.\end{aligned}$$

The Mellin Transform and the $S\alpha S$ Characteristic Function

- We now choose f to be a $S\alpha S$ density p_X, with

$$\phi_X(t) = e^{-\sigma^\alpha |t|^\alpha}.$$

- This means that

$$\begin{aligned}\mathcal{M}[p_X](\gamma) &= \int_0^\infty \zeta^{\gamma-1} p_X(\zeta) d\zeta \\ &= \frac{\Gamma(\gamma)}{2\pi} \int_{-\infty}^\infty (-it)^{-\gamma} \phi_X(t) dt\end{aligned}$$

- Using the fact that $\phi_X(t)$ is real and $\phi_X(t)^* = \phi_X(-t)$, it follows that

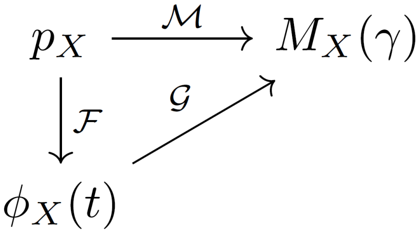
$$\mathcal{M}[p_X](\gamma) = \frac{\Gamma(\gamma)\cos\left(\frac{\gamma\pi}{2}\right)}{\pi} \int_0^\infty t^{-\gamma} \phi_X(t) dt.$$

The Fourier-Mellin Triangle

- Going a step further, we can identify

$$\int_0^\infty t^{-\gamma}\phi_X(t)dt = \mathcal{M}[\phi_X](1-\gamma)$$
$$\Rightarrow \mathcal{M}[p_X](\gamma) = \frac{\Gamma(\gamma)\cos\left(\frac{\gamma\pi}{2}\right)}{\pi}\mathcal{M}[\phi_X](1-\gamma).$$

- This all can now be summarized by the Fourier-Mellin triangle:



where

$$\mathcal{G}[\phi_X](\gamma) = \frac{\Gamma(\gamma)\cos\left(\frac{\gamma\pi}{2}\right)}{\pi}\mathcal{M}[\phi_X](1-\gamma).$$

The Mellin Transform of $S\alpha S$ Densities

- We can use the Fourier-Mellin triangle to evaluate the Mellin transform of $S\alpha S$ densities.
- In particular, we have

$$M_X(\gamma) = \frac{\Gamma(\gamma)\cos\left(\frac{\gamma\pi}{2}\right)}{\pi}\int_0^\infty e^{-\sigma^\alpha t^\alpha} t^{-\gamma} dt$$
$$= \frac{\sigma^{\gamma-1}\Gamma(\gamma)\Gamma\left(\frac{1-\gamma}{\alpha}\right)}{\pi\alpha}\cos\left(\frac{\gamma\pi}{2}\right).$$

- Note that this method generalizes to any symmetric distribution, and can also be further generalized to asymmetric distributions.
- See di Paolo and Pinnola (2012) for more details.

Recovering the Density: The Inverse Mellin Transform

- The Fourier-Mellin triangle provides a convenient way to obtain the Mellin transform.
- We can use the Mellin transform of $S\alpha S$ densities to recover the density.
- This approach has been developed in Cottone and Di Paolo (2009) and Di Paolo and Pinnola (2012).
- The relevant tool is the inverse Mellin transform:

$$p_X(x) = \frac{1}{2\pi i} \int_{\rho - i\infty}^{\rho + i\infty} M_X(\gamma) |x|^{-\gamma} d\gamma, \ x \neq 0.$$

- Condition:
  1. γ must lie in the fundamental strip.

The Fundamental Strip

- The fundamental strip is the set of $\rho = \mathrm{Re}(\gamma)$ for which the Mellin integral converges.
- To see when this occurs for the Mellin transform

$$M_X(\gamma) = \int_0^\infty p_X(x) x^{\gamma-1} dx,$$

we can use the Fourier-Mellin triangle; i.e.,

$$M_X(\gamma) = \frac{\Gamma(\gamma)\cos\left(\frac{\gamma\pi}{2}\right)}{\pi} \mathcal{M}[\phi_X](1-\gamma).$$

The Fundamental Strip

- In particular, observe that

$$|\mathcal{M}[\phi_X](1-\gamma)| \leq \int_0^1 t^{-\rho} dt + \int_1^\infty |\phi_X(t)| dt.$$

- Since

$$\int_0^\infty e^{-\sigma^\alpha t^\alpha} dt = \frac{\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\sigma^\alpha},$$

it follows that the Mellin transform converges for $0 < \rho < 1$.

Approximation via the Integral Density Representation

- The integral density representation

$$p_X(x) = \frac{1}{2\pi i}\int_{\rho - i\infty}^{\rho + i\infty} M_X(\gamma)|x|^{-\gamma} d\gamma,\ x \neq 0$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} M_X(\rho + i\eta)|x|^{-\rho - i\eta} d\eta,\ x \neq 0,$$

  lends itself to approximation.

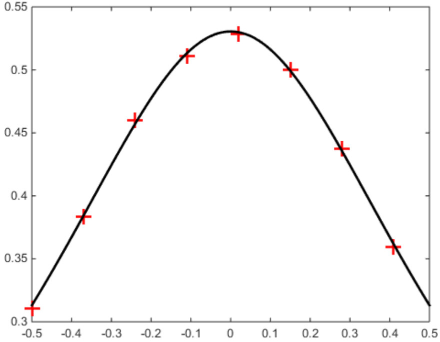
- In particular, we can use the trapezoidal approximation

$$p_X(x) \approx \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} M_X(\gamma_k)|x|^{-\gamma_k},$$

  where $\gamma_k = \rho + ik\Delta\eta$.

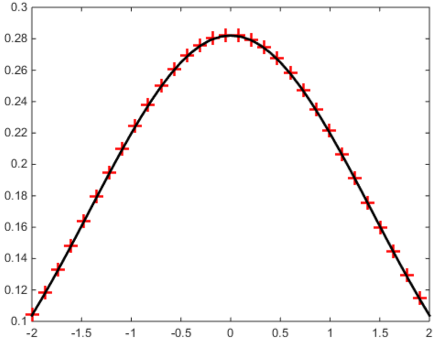
Approximation via the Integral Density Representation

- Di Paolo and Pinnola (2012) investigated the trapezoidal approximation.
- For the symmetric Cauchy density ($\alpha = 1$), they found that for $\sigma = 0.6$, choosing $\Delta\eta = 0.4$, $\rho = 0.5$ leads to



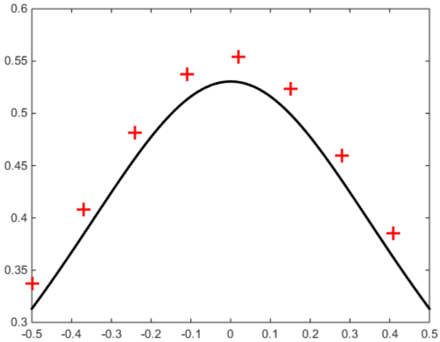
Approximation via the Integral Density Representation

- For the symmetric Gaussian density ($\alpha = 2$), Cottone and di Paolo (2009) found that for $\sigma^2 = 1$, choosing $\Delta\eta = 0.4$, $\rho = 0.4$, leads to



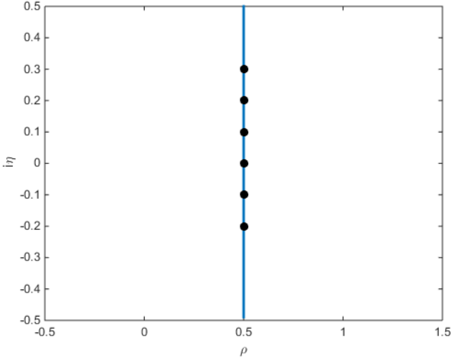
Approximation via the Integral Density Representation

- However, there is a problem when other approximation parameters are chosen.
- E.g., symmetric Cauchy with $\sigma = 0.6$, choosing $\Delta\eta = 0.4$, $\rho = 0.2$ (vs $\rho = 0.5$) leads to



Bounding the Approximation Error

- To overcome this problem, we need error bounds.
- There are two sources of error:
  1. Truncation error.
  2. Discretization error.



Bounding the Truncation Error

- For the truncation error, we need to bound the integral

$$
\begin{aligned}
|E_{T,R}| &= \left| \frac{1}{2\pi} \int_{m\Delta\eta}^{\infty} M_X(\rho + i\eta)|x|^{-\rho - i\eta} d\eta \right| \\
&\leq \frac{1}{2\pi} \int_{m\Delta\eta}^{\infty} \left| \frac{\Gamma(\rho + i\eta)\Gamma\left(\frac{1-\rho-i\eta}{\alpha}\right)}{\pi\alpha} \cos\left(\frac{(\rho + i\eta)\pi}{2}\right) |x|^{-\rho - i\eta} \right| d\eta \\
&\leq \frac{1}{2\pi^2\alpha} \int_{m\Delta\eta}^{\infty} \left| \Gamma(\rho + i\eta)\Gamma\left(\frac{1-\rho-i\eta}{\alpha}\right) \cosh\left(\frac{\pi\eta}{2}\right) \right| |x|^{-\rho} d\eta.
\end{aligned}
$$

- In the case $\alpha = 1$, we can use

$$
\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}.
$$

Bounding the Truncation Error

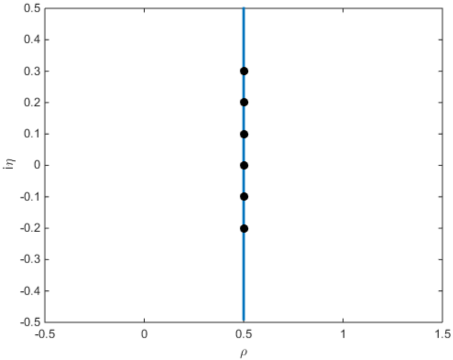
- This leads to

$$
\begin{aligned}
|E_T| &\leq \frac{|x|^{-\rho}}{2\pi}\left(\int_{m\Delta\eta}^{\infty}\left|\frac{\cosh\left(\frac{\pi\eta}{2}\right)}{\sin(\pi(\rho+i\eta))}\right|d\eta \right. \\
&\qquad \left. + \int_{-\infty}^{-m\Delta\eta}\left|\frac{\cosh\left(\frac{\pi\eta}{2}\right)}{\sin(\pi(\rho+i\eta))}\right|d\eta\right) \\
&\leq \frac{|x|^{-\rho}\sqrt{2}}{2\pi}\left(\int_{m\Delta\eta}^{\infty}\frac{\cosh\left(\frac{\pi\eta}{2}\right)}{\cosh(\pi\eta)}d\eta + \int_{-\infty}^{-m\Delta\eta}\frac{\cosh\left(\frac{\pi\eta}{2}\right)}{\cosh(\pi\eta)}d\eta\right)
\end{aligned}
$$

- Observe that for $|x| > 1$, the truncation error bound improves for larger ρ.

Bounding the Discretization Error

- We now turn to the discretization error.



Bounding the Discretization Error

- The discretization error of the trapezoidal rule can be obtained using the residue theorem.
- Define

$$I_{\Delta\eta}(x) = \frac{\Delta\eta}{2\pi} \sum_{k=-\infty}^{\infty} M_X(\gamma_k)|x|^{-\gamma_k},$$

where $\gamma_k = \rho + i\Delta\eta$.

- The discretization error is then

$$DE = |p_X(x) - I_{\Delta\eta}(x)|.$$

Bounding the Discretization Error

- Under certain regularity conditions, the trapezoidal rule has

$$DE \leq \frac{2M}{e^{2\pi c/h} - 1},$$

where c is a bound on the analytic region of the function being integrated.

- In our case, the function is

$$M_X(\gamma)|x|^{-\gamma}. \tag{6}$$

Bounding the Discretization Error

- As such,

$$DE \leq \frac{2M}{e^{2\pi(1-\rho)/\Delta\eta} - 1},$$

where

$$\int_{-\infty}^{\infty} \left| M_X(\rho + iu - r)|x|^{-\rho - iu + r} \right| du \leq M,$$

for all $\rho - 1 < r < 0$.

- A key point is that the discretization error decays as $O(e^{-2\pi(1-\rho)/\Delta\eta})$.

An Important Observation

- In both the truncation error and the discretization error, the term

$$|x|^{-\rho}$$

appears on the numerator.
- This means that the approximation *improves* for large $|x|$.
- That is, this method can be useful for approximating the tails of $S\alpha S$ densities.

More On Fractional Calculus

- Earlier in the talk the Riemann-Liouville fractional derivative was introduced as

$$(D_{\pm}^{\gamma} f)(x) = \frac{(\pm 1)^n}{\Gamma(n-\gamma)} \frac{d^n}{dx^n} \int_0^{\infty} \zeta^{n-\gamma-1} f(x \mp \zeta) d\zeta,$$

where $\gamma \in \mathbb{C}$ and $n = [\rho] + 1$, where $\rho = \mathrm{Re}(\gamma)$.

- Another type of fractional derivative is due to Riesz, given by

$$(D^{\gamma} f)(x) = -\frac{1}{2\cos(\gamma\pi/2)}((D_{+}^{\gamma} f)(x) + (D_{-}^{\gamma} f)(x)).$$

More on Fractional Calculus

- The Riesz fractional derivative has a strong link to fractional moments.
- Cottone and Di Paolo (2009) have shown that

$$(D^{\gamma}\phi_X)(0) = -\mathbb{E}[|X|^{\gamma}],\ \mathrm{Re}(\gamma) > 0.$$

- This can be viewed analogously to the usual result

$$\mathbb{E}[X^n] = i^{-n}\phi_X^{(n)}(0). \tag{7}$$

More on Fractional Calculus

- To prove it, take the Fourier transform of the Riemann-Liouville fractional derivative.
- This yields

$$\mathcal{F}[(D_{\pm}^{\gamma}\phi_X)](x) = (\mp ix)^{\gamma}\mathcal{F}[\phi_X](x),$$

  analogous to the integer derivative case.
- Taking the inverse Fourier transform and setting t to zero yields

$$(D_{\pm}^{\gamma}\phi_X)(0) = \mathbb{E}[(\mp iX)^{\gamma}].$$

More on Fractional Calculus

- The result

$$(D^{\gamma}\phi_X)(0) = -\mathbb{E}[|X|^{\gamma}], \ \mathrm{Re}(\gamma) > 0.$$

then follows by straightforward manipulations.

Fractional Moments

- A standard result is that the fractional moments for $S\alpha S$ random variables are given by

$$\mathbb{E}[|X|^p] = \frac{2^{p+1}\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(-\frac{p}{\alpha}\right)}{\alpha\sqrt{\pi}\Gamma\left(-\frac{p}{2}\right)}\sigma^p.$$

- Sometimes it is also useful to compute moments of the form:

$$\mathbb{E}[|X-\mu|^p].$$

Fractional Moments

- Why? Consider the α-stable noise channel

$$Y = X + N,$$

where $N \sim S_\alpha(\sigma, 0, 0)$.

- This is a useful model for interference in large scale wireless communication networks.
- A key step in deriving an upper bound on the capacity

$$\max_{r_X} I(X; Y)$$

is to compute moments of the form $\mathbb{E}[|X - \mu|^p]$.

Fractional Moments

- The problem of finding fractional moments $\mathbb{E}[|X - \mu|^p]$ has been studied by Matsui and Pawlas (2014) in the case $\alpha > 1$.
- Their approach relied on the use of the Marchaud fractional derivative

$$\frac{d^\gamma}{dt^\gamma} f(t) = \frac{\lambda}{\Gamma(1-\lambda)} \int_{-\infty}^{t} \frac{f^{(k)}(t) - f^{(k)}(u)}{(t-u)^{1+\lambda}} du, \;\; t \in \mathbb{R},$$

where $\gamma = k + \lambda$, with $k \in \mathbb{N}$ and $0 < \lambda < 1$.

Fractional Moments

- Let $m_{\mu,1+\lambda} = \mathbb{E}[|X - \mu|^{1+\lambda}]$.
- Matsui and Pawlas (2014) showed that for $S\alpha S$ random variables with $1 < 1 + \lambda \leq 2$

$$m_{\mu,1+\lambda} = \frac{\lambda\sigma^{1+\lambda}}{\sin\left(\frac{\lambda\pi}{2}\right)\Gamma(1-\lambda)}\left[\frac{\mu}{\sigma}\int_0^\infty u^{-(1+\lambda)}e^{-u^\alpha}\sin\left(\frac{\mu u}{\sigma}\right)du\right.$$
$$\left.+\alpha\int_0^\infty u^{\alpha-\lambda-2}e^{-u^\alpha}\cos\left(\frac{\mu u}{\sigma}\right)du\right].$$

- We want to extend this result to the case $0 < 1 + \lambda < 1$.

Fractional Moments

- To do this we use the identity of Cottone and Di Paolo (2009)

$$(D^{\gamma}\phi_X)(0) = -\mathbb{E}[|X|^{\gamma}],\ \mathrm{Re}(\gamma) > 0$$

  and the definition of the Riesz fractional derivative

$$(D^{\gamma}f)(x) = -\frac{1}{2\cos(\gamma\pi/2)}((D_{+}^{\gamma}f)(x) + (D_{-}^{\gamma}f)(x)).$$

- This means we only need to compute the individual Riemann-Liouville fractional derivatives

$$(D_{\pm}^{\gamma}f)(x) = \frac{(\pm 1)^n}{\Gamma(n-\gamma)}\frac{d^n}{dx^n}\int_0^{\infty}\zeta^{n-\gamma-1}f(x \mp \zeta)d\zeta,$$

Fractional Moments

- In our case, we need to compute

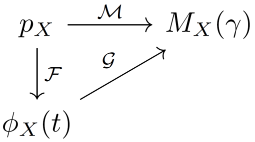
$$(D_{\pm}^{\gamma} f)(x) = \frac{\pm 1}{\Gamma(1-\gamma)} \frac{d}{dx} \int_0^{\infty} \zeta^{-\gamma} e^{i\mu t - \sigma^{\alpha}|x \mp \zeta|^{\alpha}} d\zeta,$$

- This yields

$$\mathbb{E}[|X+\mu|^p] = \frac{\sigma^p}{\Gamma(1-p)\cos(p\pi/2)} \left[\mu \int_0^{\infty} u^{-p} e^{-u^{\alpha}} \sin(\mu u/\sigma) du \right. \\ \left. + \alpha \int_0^{\infty} u^{\alpha-p-1} e^{-u^{\alpha}} \cos(\mu u/\sigma) du \right].$$

A Summary So Far

- So far for univariate $S\alpha S$ random variables, we have looked at:
  1. The link between the Mellin transform and fractional calculus.
  2. Obtained the Fourier-Mellin triangle



  3. Using the integral representation to approximate $S\alpha S$ densities.
  4. Derived fractional moments $\mathbb{E}[|X - \mu|^p]$.

Extensions to Multivariate $S\alpha S$

- It is also possible to extend a number of the results to the multivariate setting.
- This is achieved via the multivariable Mellin transform and multivariable fractional calculus.

Multivariate $S\alpha S$ Random Vectors

- The definition of $S\alpha S$ random variables can be generalized.
- I.e., a random vector $X = (X_1, \ldots, X_n)$ is $S\alpha S$ if for any $A, B > 0$, there is a $C > 0$ and vector $\in \mathbb{R}^n$ such that

$$A\mathbf{X}^{(1)} + B\mathbf{X}^{(2)} \stackrel{d}{=} C\mathbf{X} + \mathbf{d}, \tag{8}$$

and $\mathbf{X} \stackrel{d}{=} -\mathbf{X}$.

The Characteristic Function

- For $S\alpha S$ random vectors, we can write the characteristic function in two ways.
- In the bivariate case, we have

1.
$$\phi(t_1, t_2) = \int_{\mathbb{R}^2} e^{i\mathbf{t}\cdot\mathbf{x}} p(x_1, x_2) dx_1 dx_2.$$

2.
$$\phi(t_1, t_2) = \exp\left\{ -\int_{\mathbb{S}^{d-1}} \left| \sum_k t_k s_k \right|^{\alpha} d\Gamma(s_1, s_2) \right\}.$$

The Multivariate Mellin Transform

- By exploiting the link with the Fourier transform, we can generalize the Mellin transform to $\mathbb{R}^d$ as

$$\mathcal{M}[f](\mathbf{s}) = \int_0^\infty \cdots \int_0^\infty f(\zeta) \prod_{i=1}^{d} \zeta_i^{s_i - 1} d\zeta. \tag{9}$$

The Multivariate Riemann-Liouville Fractional Integral

- Similarly, we can define the multivariable Riemann-Liouville fractional integral as

$$(I_{\pm}^{\gamma} f)(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \cdots \int_0^{\infty} f(x \mp \zeta) \prod_{i=1}^{d} \zeta_i^{\gamma_i - 1} d\zeta, \tag{10}$$

which is again closely linked to the Mellin transform as

$$\Gamma(\alpha)(I_{\pm}^{\gamma} f)(\mathbf{x}) = \int_0^{\infty} \cdots \int_0^{\infty} f(x \mp \zeta) \prod_{i=1}^{d} \zeta_i^{\gamma_i - 1} d\zeta. \tag{11}$$

The Multivariate Riemann-Liouville Fractional Integral

- We can also consider the Fourier transform of $(I_{\pm}^{\gamma} f)(\mathbf{x})$, which is

$$\mathcal{F}[(I_{\pm}^{\gamma} f)(\mathbf{x})](\mathbf{t}) = \frac{\mathcal{F}[f](t)}{\prod_{j=1}^{d} (\mp it)^{\gamma_i}}. \tag{12}$$

- This yields

$$(I_{\pm}^{\gamma} f)(0) = \mathbb{E}[\prod_{j=1}^{d} (\mp iX_j)^{-\gamma_j}]. \tag{13}$$

The Multivariate Riemann-Liouville Fractional Integral

- By applying these results to the characteristic function and taking the inverse Mellin transform, we obtain

$$\phi(\pm\zeta) = \frac{1}{2\pi i}\int_{\rho+i\mathbb{R}^d} \Gamma(\gamma)\mathbb{E}[\prod_{j=1}^{d}(\mp iX_j)^{-\gamma_j}]\zeta^{-\gamma}d\gamma,\ \zeta \succ 0. \qquad (14)$$

- This yields a third way of representing the characteristic function.
- This is important because it implies that the characteristic function is completely described by the fractional moment surface

$$\mathbb{E}[\prod_{j=1}^{d}(\mp iX_j)^{-\gamma_j}] \qquad (15)$$

A Question

- In the multivariable case, there is the issue of dependence.
- This can be characterized in two ways:
  1. In the density $p(x_1, x_2)$, for instance, using a copula:

  $$p(x_1, x_2) = p_1(x_1)p_2(x_2)c(F_1(x_1), F_2(x_2)).$$

  2. In the spectral measure Γ.

A Question

How can we relate the spectral measure $\Gamma(s_1, s_2)$ to the dependence structure in the density $p(x_1, x_2)$?

A Question

- The representation

$$\phi(t_1, t_2) = \frac{1}{(2\pi i)^2} \int_{\rho + i\mathbb{R}^2} \Gamma(\gamma_1)\Gamma(\gamma_2) \times \mathbb{E}[(-iX_1^{-\gamma_1})(-iX_2^{-\gamma_2})] t_1^{-\gamma_1} t_2^{-\gamma_2} dt_1 dt_2.$$

suggests that this might be possible by studying the surface

$$\mathbb{E}[(-iX_1^{-\gamma_1})(-iX_2^{-\gamma_2})],$$

which completely characterizes the random vector.

- This remains on-going work.

Conclusions

- We looked at univariate $S\alpha S$ random variables, where we:
  1. observed the link between the Mellin transform and fractional calculus.
  2. obtained the Fourier-Mellin triangle

$$\begin{array}{ccc} p_X & \xrightarrow{\mathcal{M}} & M_X(\gamma) \\ \downarrow \mathcal{F} & \nearrow \mathcal{G} & \\ \phi_X(t) & & \end{array}$$

  3. used the integral representation to approximate $S\alpha S$ densities.
  4. derived fractional moments $\mathbb{E}[|X - \mu|^p]$.
- We then briefly showed that these ideas can be extended to the multivariable case.
- Suggested that the complex fractional moments $\mathbb{E}[(-iX_1^{-\gamma_1})(-iX_2^{-\gamma_2})]$ may be useful to help understand dependence structures.