Revisit of the resampling mechanism used in importance sampling methods

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work with R. Lamberti, Y. Petetin and F. Desbouvries

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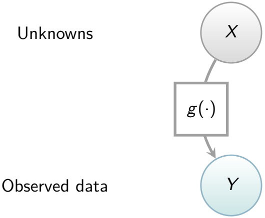
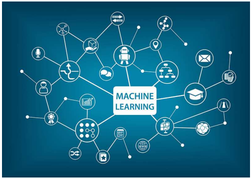
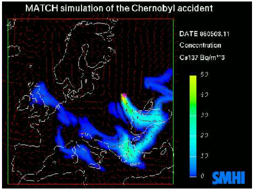
Outline

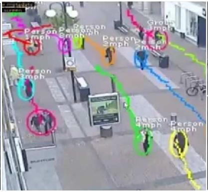
General context

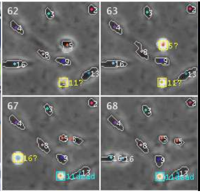
In many applications $\rightsquigarrow$ interest is in learning about unknowns from observed (noisy) data

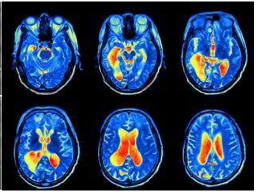












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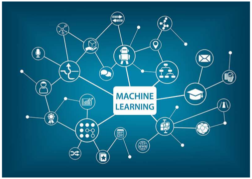

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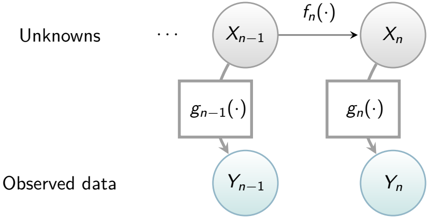

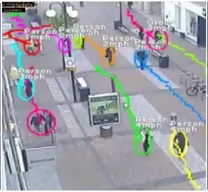

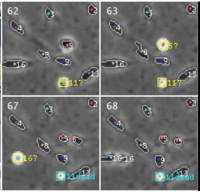

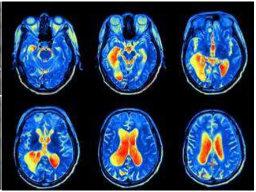

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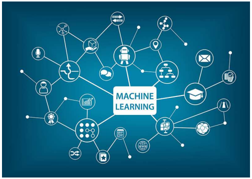

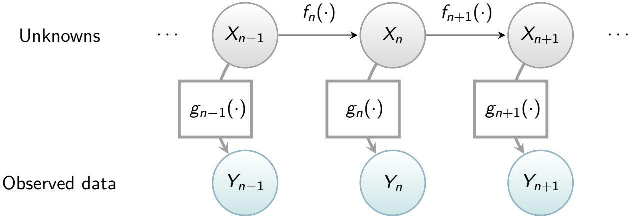

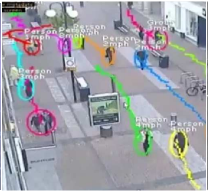

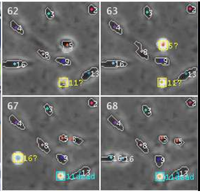

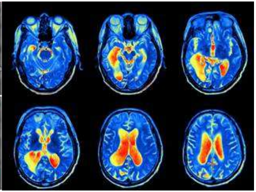

General context: Bayesian inference

Bayesian $\Rightarrow$ Instead of just a pointwise estimation of the unknowns

$$\widehat{x}_{1:n}$$

we are interested in its posterior probability density function (pdf)

$$\pi_n(x_{1:n}) \equiv p(x_{1:n}|y_{1:n}) \propto p(x_{1:n})p(y_{1:n}|x_{1:n})$$

thus characterizing all the uncertainty in the model under study.

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Unfortunately, in most cases, this posterior pdf is intractable

$\Rightarrow$ all quantities of interest such as

$$\Theta_n = \mathbb{E}_{\pi_n}[\varphi(X_n)] = \int \varphi(x_n)\pi_n(x_n)dx_n$$

cannot be computed and must be approximated

Sequential Monte-Carlo algorithms

Outline

Importance Sampling

Let us define the target distribution of interest which is known up to a normalizing constant Z:

$$\pi(x) = \frac{\gamma(x)}{Z}$$

Importance Sampling (IS) identity

For any distribution q such that $supp(\pi) \subset supp(q)$

$$\mathbb{E}_\pi[h(X)] = \int h(x) \frac{\pi(x)}{q(x)} q(x) dx$$

$q(\cdot)$ is called *importance (or proposal / instrumental) distribution*

$q(\cdot)$ can be chosen arbitrarily, in particular easy to sample from.

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$$\begin{aligned}\mathbb{E}_\pi[h(X)] &= \int h(x) \frac{\pi(x)}{q(x)} q(x) dx \\ &= \int h(x) w(x) q(x) dx = \mathbb{E}_q[h(X) w(x)]\end{aligned}$$

$q(\cdot)$ is called *importance (or proposal / instrumental) distribution*
$w(x) = \pi(x)/q(x)$ is called *importance weight*.

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$$\mathbb{E}_\pi[h(X)] = \int h(x)w(x)q(x)dx \quad \text{with } w(x) = \pi(x)/q(x)$$

- Draw independently N_p samples from $q(\cdot)$

$$\text{for } j = 1, \dots, N_p : \; X^j \overset{iid}{\sim} q(\cdot)$$

- Plugging this expression in the IS identity, we obtain [by the Law of Large numbers]:

$$\frac{1}{N}\sum_{i=1}^{N} w(X^i)h(X^i) \xrightarrow[N \to \infty]{a.s.} \mathbb{E}_\pi[h(X)]$$

or self-normalized version (used when Z is unknown)

$$\sum_{i=1}^{N} \frac{w(X^i)}{\sum_{j=1}^{N} w(X^j)} h(X^i) \xrightarrow[N \to \infty]{a.s.} \mathbb{E}_\pi[h(X)]$$

$$\text{with } w(X^i) = \frac{\gamma(X^i)}{q(X^i)}$$

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General SMC methodology

Aim: Approximate a sequence of target pdf of increasing dimension

$$\left\{\pi_n(x_{1:n})\right\}_{n\geq 1}$$

i.e. the dimension of its support forms an increasing sequence with n.

- In practice: π_n only known up to a normalizing constant,

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- Originally developed for filtering in hidden Markov model (HMM) with

$$\gamma_n(x_{1:n}) = p(x_{1:n}, y_{1:n})$$

BUT can be used for other sequence of target pdf.

How does SMC work?

- Sequence of *importance sampling* (IS) steps, where at each step n
  - the target distribution is $\pi_n(x_{1:n}) = \gamma_n(x_{1:n})/Z_n$
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Procedure at time n and $\forall j = 1, \dots, N$ [Gordon et al, 1993]

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$$W_n^j \propto W(X_{1:n-1}^j, \widetilde{X}_n^j) = \frac{\gamma_n(X_{1:n-1}^j, \widetilde{X}_n^j)}{q_n(X_{1:n-1}^j, \widetilde{X}_n^j)} = \underbrace{\frac{\gamma_{n-1}(X_{1:n-1}^j)}{q_{n-1}(X_{1:n-1}^j)}}_{\substack{W(X_{1:n-1}^j) \\ \text{Previous weight}}} \underbrace{\frac{\gamma_n(X_{1:n-1}^j, \widetilde{X}_n^j)}{\gamma_{n-1}(X_{1:n-1}^j) q_n(\widetilde{X}_n^j|X_{1:n-1}^j)}}_{\substack{\widetilde{w}(X_{1:n-1}^j, \widetilde{X}_n^j) \\ \text{Incremental weight}}}$$

$\Rightarrow$ for filtering problem in HMM with $\gamma_n(x_{1:n}) = p(x_{1:n}, y_{1:n})$, we simply have

$$\widetilde{w}(X_{1:n-1}^j, \widetilde{X}_n^j) = \frac{g_n(y_n|\widetilde{X}_n^j) f_n(\widetilde{X}_n^j|X_{n-1}^j)}{q_n(\widetilde{X}_n^j|X_{1:n-1}^j)}.$$

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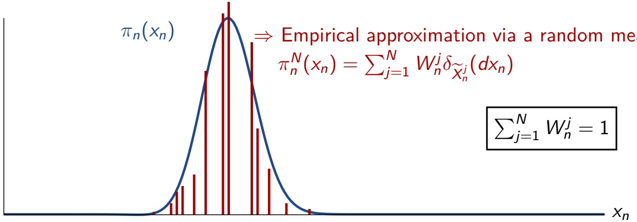

$\Rightarrow$ Empirical approximation via a random measure $\{W_n^j, \widetilde{X}_n^j\}_{j=1}^N$

$\pi_n^N(x_n) = \sum_{j=1}^N W_n^j \delta_{\widetilde{X}_n^j}(dx_n)$

$$\boxed{\sum_{j=1}^N W_n^j = 1}$$

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$$\Theta_n = \mathbb{E}_{\pi_n}[\varphi(X_n)] \Rightarrow \boxed{\widehat{\Theta}_n^{SIS,N} = \sum_{j=1}^{N} W_n^j \varphi(\widetilde{X}_n^j)}$$

Sequential Importance Sampling

Sequential Importance Sampling Algorithm

At time 1: for $j = 1, \dots, N_p$, sample $X_1^j \sim q_1(x_1)$ and set $w_1^j = \frac{\gamma_1(X_1^j)}{q_1(X_1^j)}$

for time $k > 1$ **do**

  for $j = 1, \dots, N$ **do**

    Sample $\widetilde{X}_n^j \sim q_n(\cdot|X_{n-1}^j)$

    Compute Importance weight $W_n^j \propto W_{n-1}^j \frac{\gamma_n(X_{1:n-1}^j, \widetilde{X}_n^j)}{\gamma_{n-1}(X_{1:n-1}^j) q_n(\widetilde{X}_n^j | X_{1:n-1}^j)}$

    Set $X_n^j = \widetilde{X}_n^j$

  end for

  Output Approximations :

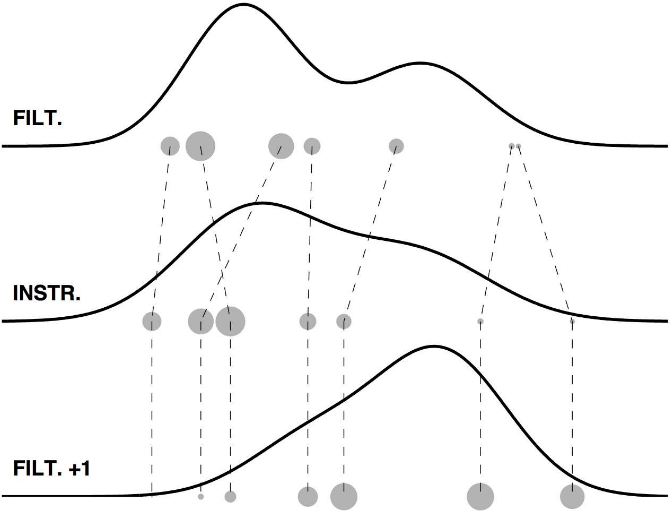
  Target pdf: $\widehat{\pi}_n(x_{1:n}) = \sum_{i=1}^N W_n^j \delta_{X_{1:n}^i}(dx_{1:n})$

  $\Theta_n = \mathbb{E}_{\pi_n}[\varphi(X_n)] \approx \widehat{\Theta}_n^{SIS,N} = \sum_{j=1}^N W_n^j \varphi(\widetilde{X}_n^j)$

  Normalizing constant: $\widehat{Z}_n = \sum_{j=1}^N W_n^i$

end for

Sequential Importance Sampling



One step of the SIS algorithm with just seven particles.

SIS : Choice of the proposal distribution

The so-called "optimal" choice of $q_n(\cdot)$ (for filtering in HMM) that minimizes the variance of the importance weights, consists in setting

$$q_n(x_n|x_{n-1}) = \frac{g_n(y_n|x_n)f_n(x_n|x_{n-1})}{\int g_n(y_n|x_n)f_n(x_n|x_{n-1})dx_n}$$

Consequently the weights does not depend on current state value but only on the previous one ($\mathbf{x}_{n-1}$), i.e. :

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This is however usually not feasible and common choices include :

- the prior $q_n(x_n|x_{n-1}) = f_n(x_n|x_{n-1})$ (and then $W_n^j \propto W_{n-1}^j g_n(y_n|X_n^j)$),
- approximations (sometimes heuristic) to the optimal one (moment matching, use of EKF or UKF, ...),
- tuning parameters of $q_n(\cdot)$ so as to maximize some criterions: effective sample size, entropy, ...

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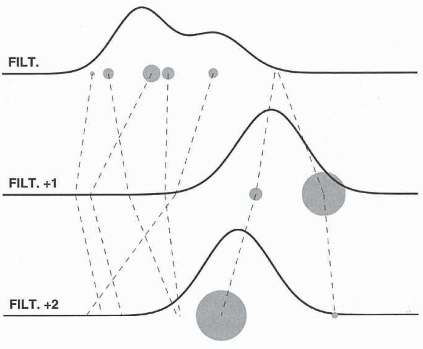
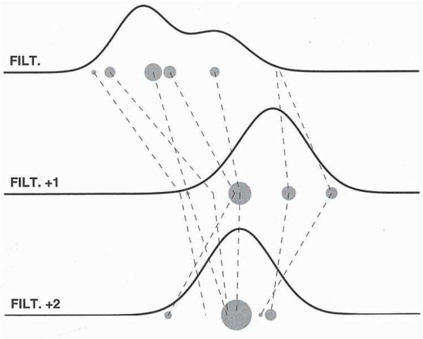
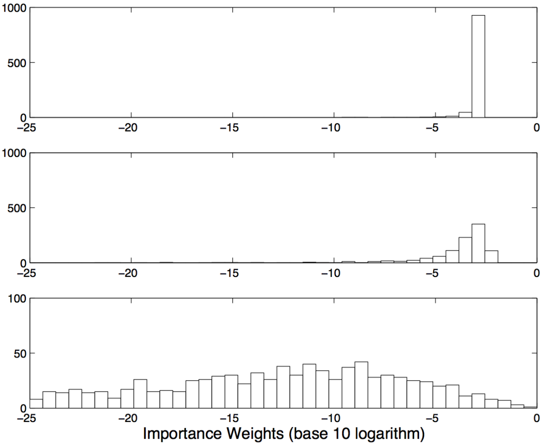




SIS with the of the prior (left) and the optimal (right) distribution as proposal.

⇒ Choice of this proposal distribution is an important step when one want to design an efficient SIS algorithm.

SIS: Weight Degeneracy



Histograms of the base 10 logarithm of the normalized weights for $t = 1$ *(top),* $t = 50$ *(middle) and* $t = 100$ *(bottom) for a simple stochastic volatility model.*

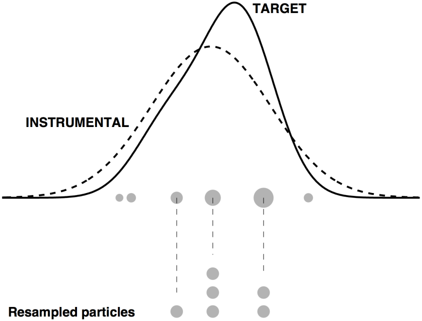
The algorithm performance collapse as time (n) increases... After a few time steps, only a very small number of particles have non negligible weigths !!

SIS: Resampling Step

Problem: After a few time steps, only a very small number of particles have non negligible weigths !

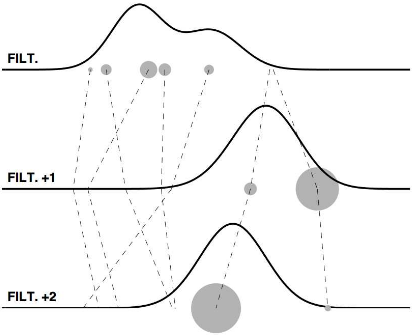
Solution: Replicate particles with large weights and eliminate those with small weights to prevent the problems we saw with SIS (at the price of a, usually moderate, increase in variance).
↪ Use random resampling techniques by taking importance weights: multinomial, residual, ...

⇒ Sequential Importance Resampling



Sequential Importance Sampling Resampling

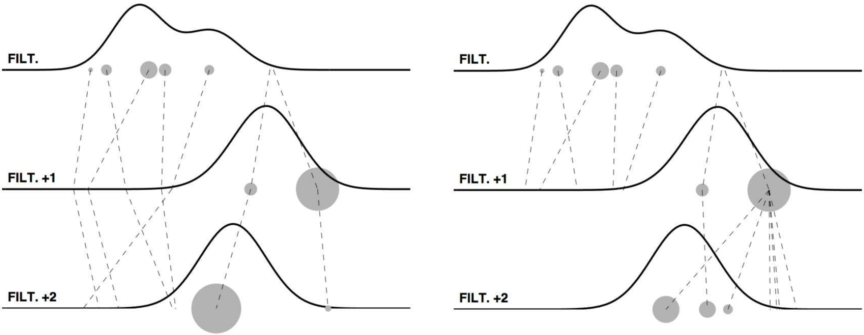
SIS (left) versus SIR (right)



The resampling step is necessary to ensure the long-term stability of the filtering algorithm.

$\hookrightarrow$ Maintain a reasonable number of contributing particles at all times.

SIS (left) versus SIR (right)



The resampling step is necessary to ensure the long-term stability of the filtering algorithm.

$\hookrightarrow$ Maintain a reasonable number of contributing particles at all times.

BUT it reduces the number of distinct samples $\Rightarrow$ *sample impoverishment*

SMC technique: summary

- Sequence of *importance sampling* (IS) steps, where at each step n
  - the target distribution is $\pi_n(x_{1:n}) = \gamma_n(x_{1:n})/Z_n$
  - the importance distribution is $q_n(x_{1:n}) = q_1(x_1)\prod_{k=2}^{n} q_k(x_k|x_{1:k-1})$

Procedure at time n and $\forall j = 1, \ldots, N$ [Gordon et al, 1993]

- **Sampling** - Propagate each trajectory: $\widetilde{X}_n^j \sim q_n(x_n|X_{1:n-1}^j)$
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$$W_n^j \propto W(X_{1:n-1}^j, \widetilde{X}_n^j) = W(X_{1:n-1}^j)\widetilde{w}(X_{1:n-1}^j, \widetilde{X}_n^j)$$

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- **Resampling** (optional but necessary to avoid *weight degeneracy*): Sample N times from the random measure $\{W_n^j, X_{1:n-1}^j, \widetilde{X}_n^j\}_{j=1}^N$ to obtain $\{W_n^j = \frac{1}{N}, X_{1:n}^j\}_{j=1}^N$

$$\boxed{\widehat{\Theta}_n^{SIR,N} = \frac{1}{N}\sum_{j=1}^{N} \varphi(X_n^j)}$$

Outline

Effect of the (optional) multinomial resampling step:

Fights against weight degeneracy but no local benefits:

- dependency among the resampled points $\Rightarrow$ support shrinkage
- $\text{var}(\widehat{\Theta}_n^{SIR,N}) \geq \text{var}(\widehat{\Theta}_n^{SIS,N})$

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Key idea: *Ph.D thesis of R. Lamberti - [Lamberti et al, IEEE TSP 2017]*

Revisit the complete scheme
(sampling, weighting and resampling)

$\downarrow$

Benefits of the resampling mechanism
without local impoverishment of the resulting MC approx.

Outline

Characterization of sampling / weighting / resampling step

Proposition 1

Given $\{X^i_{0:n-1}\}_{i=1}^N$, the resampled particles X^j_n are identically distributed **(but dependent)** according to a pdf $\widetilde{q}^N_n(x)$, with

$$\widetilde{q}^N_n(x) = \sum_{i=1}^N q^i_n(x) h^i_n(x) \quad (1)$$

where

$$\begin{aligned} q^i_n(x) &= q(x|X^i_{0:n-1}) \\ h^i_n(x) &= \int \int \frac{\frac{\pi^i_n(x)}{q^i_n(x)}}{\frac{\pi^i_n(x)}{q^i_n(x)} + \sum_{l \neq i} \frac{\pi^l_n(x^l)}{q^l_n(x^l)}} \prod_{l \neq i} q^l_n(x^l) dx^l \end{aligned} \quad (2)$$

This dependency results in support shrinkage since, by construction, an intermediate sample can be resampled several times.

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Proposition 1

Given $\{X^i_{0:n-1}\}_{i=1}^N$, the resampled particles X^j_n are identically distributed **(but dependent)** according to a pdf $\widetilde{q}^N_n(x)$, with

$$\widetilde{q}^N_n(x) = \sum_{i=1}^N q^i_n(x) h^i_n(x) \quad (1)$$

where

$$q^i_n(x) = q(x|X^i_{0:n-1})$$
$$h^i_n(x) = \int\int \frac{\frac{\pi^i_n(x)}{q^i_n(x)}}{\frac{\pi^i_n(x)}{q^i_n(x)} + \sum_{l\neq i} \frac{\pi^l_n(x^l)}{q^l_n(x^l)}} \prod_{l\neq i} q^l_n(x^l) dx^l \quad (2)$$

Proposition ⇒ Scheme to produce independent samples from $\widetilde{q}^N_n(x)$

Proposed independent resampling

Method to sample N **independent** particles from $\widetilde{q}_n^N(x)$

$\Rightarrow$ **weighting & resampling as compound importance distrib.**

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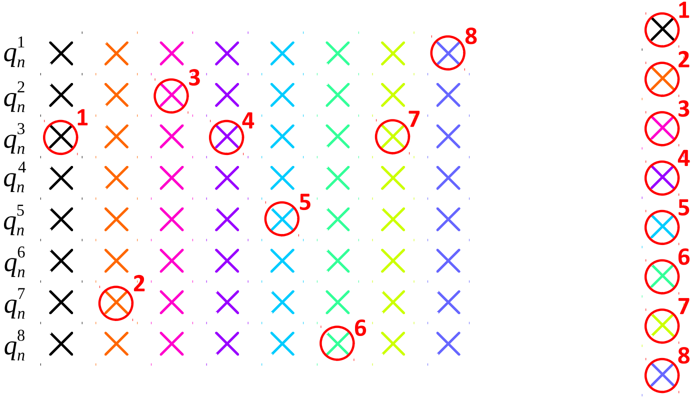

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- Sampling N independent particles from $\widetilde{q}_n^N(x)$
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  1. Equal weights: $W_n^i = 1/N$ - what is actually done after classical resampling
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  1. Equal weights: $W_n^i = 1/N$ - what is actually done after classical resampling
     $\Rightarrow$ Algorithm I-SIR
  2. IS principle: $W_n^i \propto \pi_n^i(X_n^i) \Big/ \widetilde{q}_n^N(X_n^i)$
     $\Rightarrow$ Algorithm I-SIR-w

     However $\widetilde{q}_n^N(X_n^i) = \sum_{i=1}^N q_n^i(x) h_n^i(x)$ must be approximated

     since $h_n^i(x) = \int \int \frac{\frac{\pi_n^i(x)}{q_n^i(x)}}{\frac{\pi_n^i(x)}{q_n^i(x)} + \sum_{l \neq i} \frac{\pi_n^l(x^l)}{q_n^l(x^l)}} \prod_{l \neq i} q_n^l(x^l) dx^l$ cannot be evaluated

     $\Rightarrow$ h_n^i approximated by the N^2 samples

Proposed independent resampling

Contributions $\mapsto$ Theoretical study of $\widehat{\Theta}_n^{\mathrm{I-SIR},N}$

- Mean/Variance for finite N (number of samples)

$$\mathbb{E}(\widehat{\Theta}_n^{\mathrm{I-SIR},N}|\{X_{0:n-1}^i\}_{i=1}^N) = \mathbb{E}(\widehat{\Theta}_n^{\mathrm{SIR},N}|\{X_{0:n-1}^i\}_{i=1}^N) = \mathbb{E}(\widehat{\Theta}_n^{\mathrm{SIS},N}|\{X_{0:n-1}^i\}_{i=1}^N),$$

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$$\mathrm{var}(\widehat{\Theta}_n^{\mathrm{I-SIR},N}|\{X_{0:n-1}^i\}_{i=1}^N) = \mathrm{var}(\widehat{\Theta}_n^{\mathrm{SIR},N}|\{X_{0:n-1}^i\}_{i=1}^N) - \frac{N-1}{N}\mathrm{var}(\widehat{\Theta}_n^{\mathrm{SIS},N}|\{X_{0:n-1}^i\}_{i=1}^N).$$

  - The proposed algorithm ($\widehat{\Theta}_n^{\mathrm{I-SIR}}$) would outperform the classical SIR estimate
  - Gain depends on $\mathrm{var}(\widehat{\Theta}_n^{\mathrm{SIS},N}|\{X_{0:n-1}^i\}_{i=1}^N)$

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- CLT for a single step

$$\sqrt{N}(\widehat{\Theta}^{\mathrm{I-SIR},N} - \Theta) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \underbrace{\mathrm{var}_\pi(\varphi(x))}_{\sigma_\infty^{2,\mathrm{I-SIR}}(q)}\right).$$

for comparison:

$$\sigma_\infty^{2,\mathrm{SIS}}(q) = \mathbb{E}_q\left(\frac{\pi^2(x)}{q^2(x)}(\varphi(x) - \Theta)^2\right),$$

$$\sigma_\infty^{2,\mathrm{SIR}}(q) = \mathrm{var}_\pi(\varphi(x)) + \mathbb{E}_q\left(\frac{\pi^2(x)}{q^2(x)}(\varphi(x) - \Theta)^2\right)$$

  - Asymp. variance: $\sigma_\infty^{2,\mathrm{I-SIR}}(q) \leq \sigma_\infty^{2,\mathrm{SIR}}(q)$
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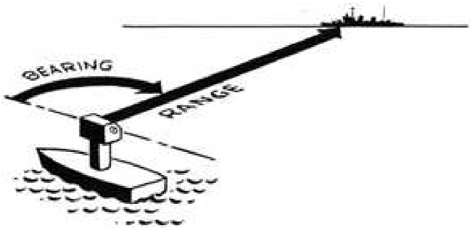
  - $\sigma_\infty^{2,\mathrm{I-SIR}}(q)$ no longer depends on the proposal distribution q
  - Difficult to compare $\sigma_\infty^{2,\mathrm{I-SIR}}(q)$ and $\sigma_\infty^{2,\mathrm{SIS}}(q)$

Outline

Numerical simulations - Tracking

- State of interest $x_n = [p_{x,n}, \dot{p}_{x,n}, p_{y,n}, \dot{p}_{y,n}]^T$ position and velocity of a target
- Tracking scenario with range-bearing measurements $\Rightarrow$ likelihood pdf:

$$g_n(y_n|x_n) = \mathcal{N}\left(y_n; \begin{bmatrix} \sqrt{p_{x,n}^2 + p_{y,n}^2} \\ \arctan \frac{p_{y,n}}{p_{x,n}} \end{bmatrix}, R\right) \text{ with } R = \begin{pmatrix} \sigma_\rho^2 & 0 \\ 0 & \sigma_\theta^2 \end{pmatrix}$$



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- Prior knowledge on the dynamics of the unknown state:

Near Constant Velocity (NCV) model

$$f_n(x_n|x_{n-1}) = \mathcal{N}(x_n; Fx_{n-1}, Q)$$

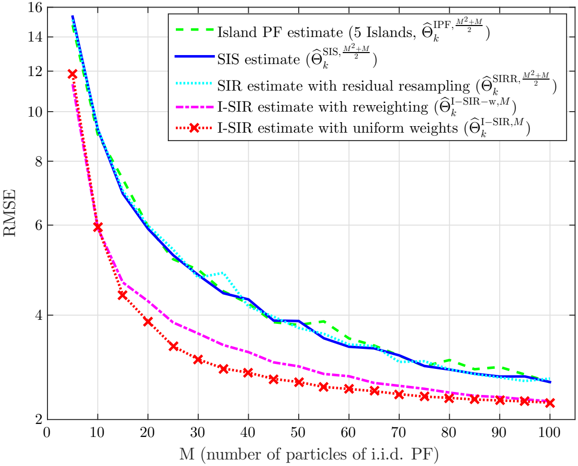
with

$$F = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } Q = \sigma_Q^2 \begin{pmatrix} \frac{1}{3} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 1 \end{pmatrix}$$

Numerical simulations - Tracking

Target tracking scenario: range-bearing measurements ($\sigma_Q = \sqrt{10}$, $\sigma_\rho = 0.05$, $\sigma_\theta = \frac{\pi}{3600}$)

Comparison with other SMC algorithms in which the number of particles is chosen so that they have the same computational cost



For RMSE=2.7

$N_{\mathrm{I-SIR}} = 50$

vs

$N_{\mathrm{SIS}} = 5050$

Proposed independent resampling

Summary of the proposed I-SIR

- No support degeneracy: better particle diversity for the next iteration
- Gain in terms of variance of the resulting estimator

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  - Resampling is not necessarily needed at each iteration
  - Independent resampling can be parallelized
  - In some cases [as in before], performs better even when $N_{\text{I}-SIR}^2 + N_{\text{I}-SIR} = 2N_{SIS}$

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Can we propose a general framework that will include classical and independent as a special case ?

Outline

Classical vs Independent

Classical
SIR

Independent
I-SIR

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Classical
SIR

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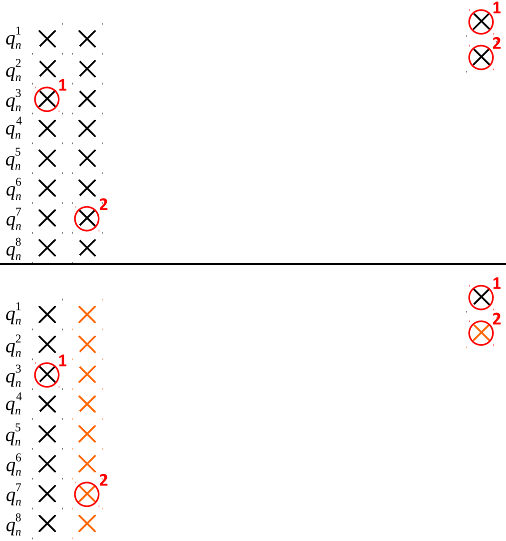

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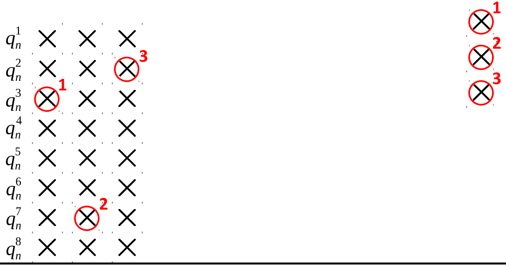

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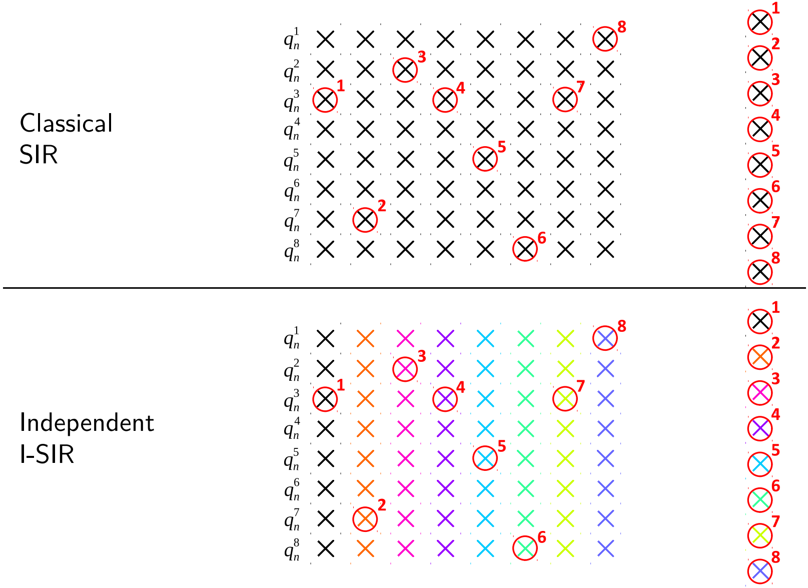

Idea $\mapsto$ Regenerate only $0 \leq k \leq N = 8$ samples per iterations

The semi-independent algorithm

Independent
I-SIR

Semi-Independent
SR

Here - $k = 3$ uniformly chosen samples are redrawn from the previous population

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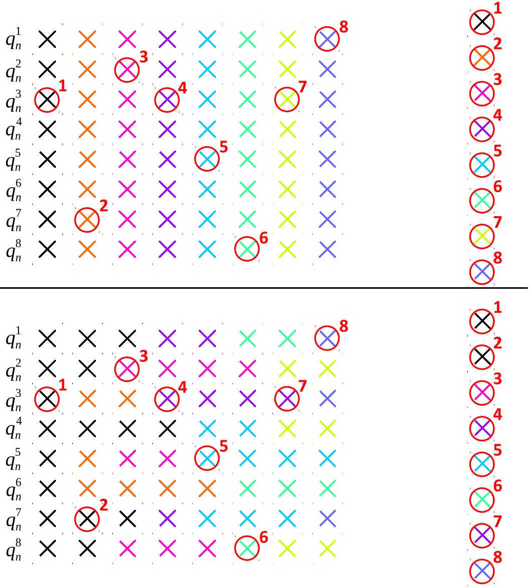

Remark: Not anymore parallelized when $0 < k < N$!

The semi-independent algorithm

Semi-Independent SR

Semi-Independent NSSR

NSSR is a fully parallelized version of this semi-independent resampling

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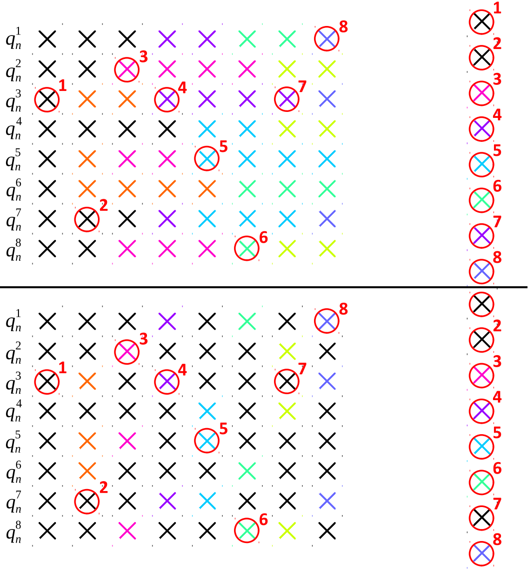

NSSR is a fully parallelized version of this semi-independent resampling

Semi-independent algorithm - Theoretical analysis

- **Generalization** of classical SIR and the previous independent I-SIR
  - When $k = 0$ regenerated samples $\Rightarrow$ SR=NSSR=SIR
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- **Statistical properties:** Given the previous set of particles $\{X_{0:n-1}^i\}_{i=1}^N$, for all k, $0 \leq k \leq N$, we have:

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Each sample X_n^i are marginally drawn from the same distribution $\widetilde{q}_n^N(x)$

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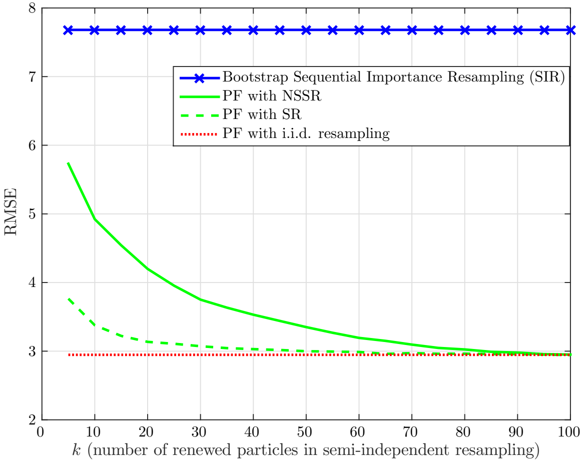
Numerical simulations - Tracking

Target tracking scenario: range-bearing measurements $(\sigma_Q = \sqrt{10},\ \sigma_\rho = 0.1,\ \sigma_\theta = \frac{\pi}{1800})$

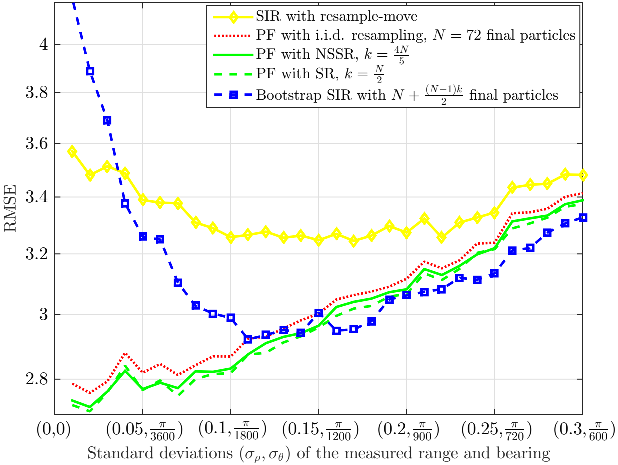


- Improvement when k increases
- note that $\widehat{\Theta}_t^{\mathrm{SR},N,k}$ (resp. $\widehat{\Theta}_t^{\mathrm{NSSR},N}$) has almost the same performance as $\widehat{\Theta}_t^{\mathrm{I-SIR},N}$ when $k \geq N/2$ (resp. $k \geq 4N/5$)

Numerical simulations - Tracking

Target tracking scenario: range-bearing measurements ($\sigma_Q = \sqrt{10}$)

Comparison with other algorithms, each having the same computational cost



- Significant gain for informative models , i.e. small values for $(\sigma_\rho, \sigma_\theta)$

Conclusion and Perspectives

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- Revisit of sampling/resampling as a compound IS distribution
  - Proposition of a general technique to draw (semi-)independent samples
  - Theoretical analysis showing the benefit of such an approach
  - At equivalent cost, the proposed approach outperforms the existing techniques in highly informative/high-dimensional models

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Perspectives

- Optimization of k (# regenerated samples): trade-off performance & cost
- Non-uniform selection of samples to regenerate