Explicit Solutions to Correlation Matrix Completion Problems in Risk and Insurance

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Overview

- We derive explicit solutions to the problem of completing a partially specified correlation matrix.
  - *Our formulas facilitate practical real world applications in large incomplete dependence structures.*
- Among the many possible completions we focus on the one with **maximal determinant.**
  - *Previously no solutions available in the form of explicit matrix expressions that are readily translated into code.*
- Our solutions are useful for testing more general algorithms for the maximal determinant correlation matrix completion problem.
- Solutions derived for **several block structures** for the locations of the unspecified entries.

Overview

- Often missing values in a set of variables lead to the construction of an **approximate correlation matrix** that lacks **definiteness**
  - *not a true correlation matrix*.
- **Nearest correlation matrix:** replacing the approximate correlation matrix by the projected nearest correlation matrix, see [Borsdorf R, Higham NJ. 2010] and [Qi H, Sun D. 2006]

We are concerned with problems in which the **missing values are in the correlation matrix itself.**

- Some of the matrix entries are known, having been:
  - *estimated*;
  - *prescribed by regulations*; or
  - *assigned by expert judgement*,

  however, the **other entries are unknown!**

Nearest correlation matrix solutions will preclude this important case as the projections required distort all elemts!

Overview

- The aim is to fill in the missing entries in order to produce a correlation matrix
- **Of course there are, in general, many possible completions!!!**

For example, the partially specified matrix

$$A = \begin{bmatrix} 1 & a_{12} \\ a_{12} & 1 \end{bmatrix}$$

is a correlation matrix for any a_{12} such that $|a_{12}| \leq 1$.

- **Our focus is on the completion with maximal determinant.**
  - given by $a_{12} = 0$ in this example.

It is always unique when completions exist!

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Bank and Insurance Context

An insurance company or a bank with many lines of business must satisfy certain capital requirements i.e. calculation of **Tier I capital (Basel III) and SCR (Solvency II)**.

- Such quantification of capital requires aggregation of multiple stochastic risk drivers over multiple lines of business.

- **PROBLEM:** *often institutions have incomplete knowledge of the underlying correlation structures between risk exposures/drivers between the different lines of business.*

Insurance and risk management utilises correlation matrices in the aggregation of risk exposures.

- **Correlations can provide risk diversification benefits in capital risk measure charges!**

Consider coherent risk measures ([Artzner et al., 1999]).

Definition (A coherent risk measure)

A coherent risk measure, $\varrho[X]$, is defined to have the following properties for any two random variables X and Y:

- Translation invariance: for any constant c, $\varrho[X + c] = \varrho[X] + c$;
- Monotonicity: if $X \leq Y$ for all possible outcomes, then $\varrho[X] \leq \varrho[Y]$;
- **Subadditivity**: $\varrho[X + Y] \leq \varrho[X] + \varrho[Y]$;
- Positive homogeneity: for any positive constant c, $\varrho[cX] = c\varrho[X]$.

We are interested in subadditivity arising from dependence!

A popular class of coherent risk measures is the so-called spectral risk measures: eg. VaR, ES and SRM.

Definition (Value-at-Risk)

The VaR of a random variable $X \sim F(x)$ at the α-th probability level, $\mathrm{VaR}_\alpha[X]$, is defined as the α-th quantile of the distribution of X, i.e.

$$\begin{aligned}\mathrm{VaR}_\alpha[X] &= F^{-1}(\alpha) = \inf\{x : \mathbb{P}\mathrm{r}[X > x] \leq 1 - \alpha\} \\ &= \inf\{x : F(x) \geq \alpha\} \\ &= \sup\{x : F(x) < \alpha\}.\end{aligned}$$

That is, VaR is the minimum threshold exceeded by X with probability at most $1 - \alpha$.

Why do we care about correlation in this problem?

- Consider a collection of risks $X_1, \ldots, X_n$ with aggregate risk measure $\varrho[\cdot]$ and individual risk capital denoted by $\varrho_i = \varrho[X_i]$.
- If these risks are combined into one business, then the total capital (coherent risk measures) for the business satisfies
$$\varrho[X_1 + \cdots + X_n] \leq \varrho_1 + \cdots + \varrho_n$$
- **Dependence between loss processes can cause increases or decreases in aggregate capital!**

Lets explore the role of dependence in sub-additive risk measures further for partial sums in two simple cases:

- Indpendent losses and
- Dependent losses

More general detailed results are discussed in [Peters and Shevchenko, 2016 Chapter 7].

- In light tailed loss distribution cases Sub-additivity of risk measures known to hold in general
  $\Rightarrow$ we can concentrate on some heavy tailed loss process examples (R.V. or Sub-exponential).

Lemma (Convolution Root Closure of Sub-exponential Distributions)

Assume that the partial sum $Z_n = \sum_{i=1}^{n} X_i$ is regularly varying with index $\rho \geq 0$, such that $\forall t > 0$:

$$\lim_{x \to \infty} \frac{F_{Z_n}(t\, x)}{F_{Z_n}(x)} = t^{\rho}$$

with each X_i being i.i.d. with positive support.

Then for all $i \in \{1, \ldots, n\}$, the X_i's are regularly varying, also with index ρ and the following asymptotic equivalence as $x \to \infty$ holds

$$\mathbb{P}\mathrm{r}\left[Z_n > x\right] \sim n \mathbb{P}\mathrm{r}\left[X_1 > x\right], \ \forall n \geq 1.$$

In [Danielsson et al, 2005] they showed the following proposition (for asset returns - negative support case)

Theorem

Consider two asset return random variables X_1 and X_2 having jointly regularly varying nondegenerate tails with tail index $\alpha > 1$.

Then VaR is subadditive in the tail region.

- **Note that X_1 and X_2 are allowed to be dependent.**

$\Rightarrow$ at sufficiently low probability levels, the VaR of a portfolio position is lower than the sum of the VaRs of the individual positions, if the return distribution exhibits fat tails!

- Examples: a multivariate Student-t distribution with degrees of freedom larger than 1 is one such example.

Conversely - we see diversification is lost in general for super heavy tails $\alpha < 1$.

Capital Quantification via Risk Measures

Common dependent loss r.v. case: based on ideas in [Asmussen, 2008]

Theorem (Partial Sums: LogNormal-Gaussian Copula)

Consider a partial sum of n losses with marginal distribution $X_i \sim F_{X_i}$ given by a LogNormal distribution satisfying the tail asymptotic given by

$$\overline{F}_{X_i}(x; \mu_i, \sigma_{ii}) \sim \frac{\sqrt{\sigma_{ii}}}{\sqrt{2\pi}\ln x} \exp\left(-\frac{(\ln x - \mu_i)^2}{2\sigma_{ii}}\right)$$

and a copula distribution C given by the multivariate Gaussian copula with mean vector $(\mu_1, \mu_2, \ldots, \mu_n)$ and covariance matrix $(\sigma_{ij})_{n\times n}$.
In this case, one has a partial sum tail asymptotic given by

$$\mathbb{P}\mathrm{r}[Z_n > x] \sim m_n \overline{F}(x; \mu_*, \sigma_*)$$

with

$$\sigma_* = \max_{1\le k\le n} \sigma_{kk}, \quad \mu_* = \max_{k:\sigma_{kk}=\sigma_*} \mu_k$$
$$m_n = \mathit{Cardinality}\{k : \sigma_{kk} = \sigma_*, \mu_k = \mu_*\}.$$

$m_n < n$ **asymptotically gives sub-additivity in risk measures.**

Sub-additivity of risk measures and **diversification benefits** of aggregated dependent loss processes can reduce or increase capital!

- *Correlation completion methods can strongly affect the outcome*

$\Rightarrow$ **regulators and industry require guidance on mathematical best practice to avoid moral hazard in artificial capital reduction!**

Without mathematical solutions for most onerous completions the regulator and the industry will generally diverge on their correlation best estimates on missing components in their BU/RT matrices!

- Regulator: increased stability
  $\Rightarrow$ **increased capital**;
- Industry: reduced cost and greater liquidity of assets
  $\Rightarrow$ **reduced capital.**

Capital Quantification via Risk Measures

The true total capital for d loss processes that are dependent is the risk measure of $\rho\left(\sum_{i=1}^{d} X_i\right)$

$\Rightarrow$ difficult to find general closed form properties of distribution of $\sum_{i=1}^{d} X_i$ in **dependent case!**

SOLVENCY II Standard Formula: approximation for total capital

- aggregate the capital requirement for different risk exposures as follows:

$$\rho\left(\sum_{i=1}^{d} X_i\right) \approx \sqrt{\boldsymbol{\rho}^T \Sigma \boldsymbol{\rho}}$$

  - $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_n]$ is a vector of capital requirements for the individual risk categories; and
  - Σ is the correlation matrix specifying the dependence: specified or constrained by the regulations.
- *Approximation based on assumption that underlying distribution of risk capital is multivariate normal, or more generally elliptically contoured.*
  the isolines are given by: $\sqrt{\boldsymbol{\rho}^T \Sigma \boldsymbol{\rho}} = c$

CHALLENGE: often in practice not all of the entries in the correlation matrix Σ are known.

- Correlation coefficients are typically fully specified in the business unit with the BU-specific risk.
- Correlations are also specified between similar risk families in different business units.

For example consider the case of just two business units BU_1 and BU_2:

- Both are exposed to risks x and y, but only BU_1 is exposed to risk z.
- Correlations are specified between risk z, x, and y in BU_1, but not between x and y in BU_2, and z in BU_1.

Correlations		*x*	*y*	*z*	*x*	*y*
BU_1	x	*1*	0.7	0.85	0.85	0.75
	y	*0.7*	1	0.6	0.5	0.85
	z	*0.85*	0.6	1	*	*
BU_2	x	*0.85*	0.5	*	1	0.75
	y	*0.75*	0.85	*	0.75	1

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Application Context

- Correlations likely to be missing in areas of risk management and insurance where data and loss event history is scarce $\Rightarrow$ **large gaps in the data records**:
  - in operational risk,
  - reinsurance,
  - catastrophe insurance,
  - life insurance, and
  - cyber risk.
- The estimation of missing correlations is also important in banking capital calculations
  - Example in the internal model-based approach to market risk and the advanced measurement approach (AMA) and (SMA) for operational risk.
- Other important applications include correlation effects in **stress testing** and **scenario analysis**

Banking Risk: Business Lines and Risk Types

Banking has three main risk classes:

- Market,
- Credit and
- Operational Risk
    - *Operational Risk is evolving, by loss events incurred, to be the leading risk type in banking out of the three core risks!*

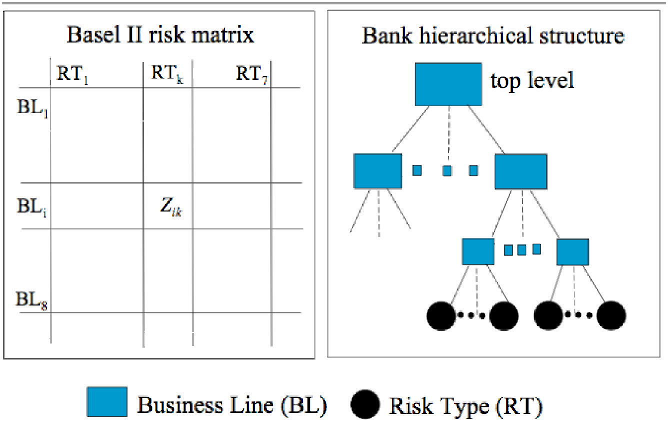
In Operational Risk

- At level 1: Basel II/III requires 56 business unit/risk type loss processes.
- At level 2 and greater granularity: this can reach 100's to 1000's of BuRT cells in practice.

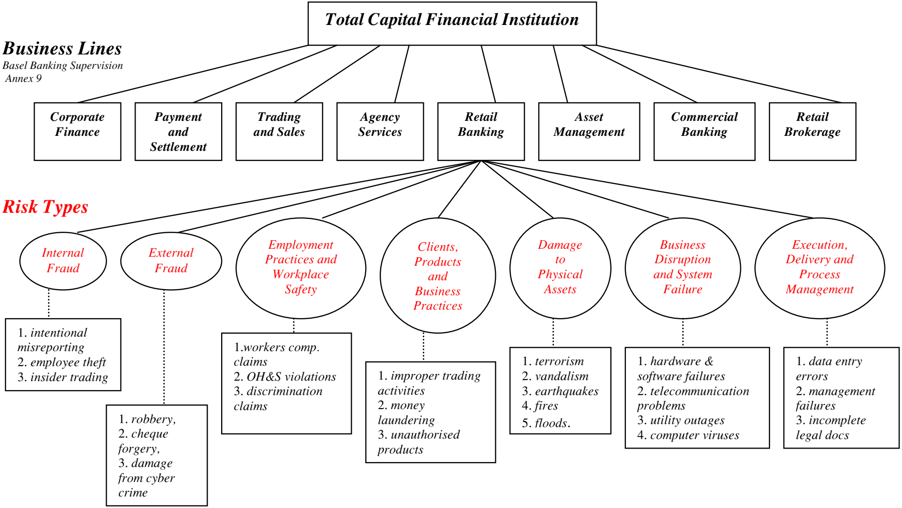
Many unknown/missing correlations present!

Banking Risk: Business Lines and Risk Types

- Advanced Measurement Approaches (AMA): Internal model for 56 risk cells (7 event types $\times$ 8 business lines).



Banking Risk: Business Lines and Risk Types



Lines of Insurance: *one can generally classify insurance companies by the type of insurance policies they write.*

- Insurance coverages are often broken down via lines of insurance.
- Information about premiums and losses is frequently analyzed by line of insurance at the company level.

Four Major Lines of Insurance:

- **Property**;
- **Casualty**;
- **Life**;
- **Health and Disability**;

Many large companies write all lines of insurance.

Insurance Risk: Business Lines and Risk Types

Each of the four major categories of insurance can be further subdivided into:

- **Personal**
  - *Personal lines are property-casualty coverages that protect an individual or family.*
- **Commercial**
  - *Commercial lines are coverages designed for businesses.*

e.g. of commercial lines of business

- professional indemnity;
- product liability;
- political risk;
- financial institutions;
- commercial auto insurance;
- workers compensation insurance;
- federal flood insurance;
- aircraft insurance;
- ocean marine insurance;

Why might there be missing or uncertain correlations between these commercial lines of business and the risk types they are exposed to ?

- Reason 1: Contract Writing Specificities!
- Reason 2: Specialty & Nature of Insured Risk!

Example: Reason 1 Most if not all commercial lines share certain similarities, however it is not unusual that each policy will be tailored for the type of business being covered and the clients unique needs!

- e.g. *structural engineering firm takes professional liability insurance to protect against claims of:*
  - *negligence in creating a buildings plans, performing inspections, and supervising construction, (project specific risks)*
  - *failure to render professional services.*
  - *specific additional coverage for each project, plus coverage for punitive damages can be added on a general cover.*

Insurance Risk: Business Lines and Risk Types

Example: Reason 2 *Specialty Types of Commercial Lines Insurance*:

- Debris Removal Insurance: *removing debris after a catastrophic events e.g. fires.*
- Builder's risk insurance: *insures buildings while they are being constructed.*
- Glass Insurance: *covers broken windows in a commercial establishment.*
- Business Interruption Insurance: *lost income and expenses resulting from property damage or loss. e.g. fire forces closure for few months, this insurance covers salaries, taxes, rents, and net profits that would have been earned.*

Very challenging to assess / estimate correlations between loss processes in such specialty risk classes!

Application Summary

In banking and insurance applications there are many business units with many BU-specific risks as well as different numbers of risk families

- Correlation matrices for each BuRT can have hundreds of columns!
- Many of the correlations between diverse BuRT's are completely unknown!

We want to complete the partial correlation matrix $\bar{\Sigma}$ to a fully specified correlation matrix; that is, since the diagonal is fully specified as ones, to a positive definite matrix.

- Many completions are possible $\Rightarrow$ *introduces uncertainty around the range of potential capital outcomes*!
- Completion of most interest is usually a best-estimate completion in some sense.
- *A good candidate is that completion which has maximum determinant*!

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Correlation Completion via MaxDet

MaxDet has several useful theoretical properties:

1. ***Existence and uniqueness***: if positive semidefinite completions exist then there is exactly **one MaxDet completion** [Grone et al., 84].
2. ***Maximum entropy model***: **MaxDet is the maximum entropy completion for the multivariate normal model**, where maximum entropy is a principle of favouring the simplest explanations. In the absence of other explanations, we should choose this principle for the null hypothesis in Bayesian analysis [Good,63].
3. ***Maximum likelihood estimation***: **MaxDet is the maximum likelihood estimate of the correlation matrix** of the unknown underlying multivariate normal model.
4. ***Analytic center***: **MaxDet is the analytic centre of the feasible region described by the positive semidefiniteness constraints**, where the analytic centre is defined as the point that maximizes the product of distances to the defining hyperplanes [Vandenberghe et al, 98].

Correlation Completion via MaxDet

When considering MaxDet for a correlation matrix we can observe the following upper bound.

- The determinant of a correlation matrix is at most 1 via Hadamard's inequality.

  Let matrix $A = [a_{ij}]$ be an $n \times n$ positive definite matrix. Then:

  $$\det A \leq a_{11} \; \cdots \; a_{nn}$$

  with equality iff A is diagonal.

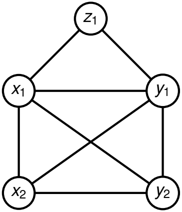
[Grone et al., 84] showed a partially specified Hermitian matrix with specified positive diagonal entries and positive principal minors (where specified) can be completed to a positive definite matrix regardless of the values of the entries

- ***iff the undirected graph of the specified entries (ignoring the leading diagonal) is chordal.***

Correlation Completion via MaxDet

- *A graph is chordal if every cycle of length ≥ 4 has a chord, which is an edge that is not part of the cycle but connects two vertices of the cycle.*
- If the graph is not chordal, then whether a positive symmetric definite completion exists depends on the specified entries.
- All the sparsity patterns considered in this work are chordal $\Rightarrow$ a positive definite symmetric completion is possible!!!

Adjacency graph for the case in previous 2 BU example below:



[Grone et al. 84] show that if a positive definite completion exists then there is a unique matrix in the class of all positive definite completions whose determinant is maximal.

Correlation Completion via MaxDet

Dealing with large matrices with block patterns of specified and unspecified entries, it is convenient to introduce the definition of a "block chordal" graph.

Block Chordal Graphs:

- A block is a subgraph which is complete in terms of edges (a clique).
- Two blocks are connected by an edge if every vertex has an edge to every other vertex, so the two blocks considered together also form a clique.
- A graph is block chordal if every cycle of blocks of length ≥ 4 has a chord.
- Finally, a block chordal graph is also chordal since every block is either fully specified or fully unspecified, so collapsing each block into one node means that we do not lose any information in the graph.

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MaxDet and the Dual Covariance Selection

[Dempster, 72] proposed a related problem known as **covariance selection**

- [Dahl et al, 08] and [Vandenberghe et al, 98] show that MaxDet completion and covariance selection are duals of each other.

Covariance selection: aims to simplify the covariance structure of a multivariate normal population by ***setting elements of the inverse of the covariance matrix to zero***.

- The statistical interpretation is that ***certain variables are set to be pairwise conditionally independent.***

Multivariate normal setting:

- *Conditional independence in general:* partition a multivariate normal random variable X into two sets: I and J (the idea being that the I variables are independent of each other, conditioning on J).
- The conditional distribution of X_I given X_J is with covariance matrix

$$\Sigma_{I|J} = \Sigma_{II} - \Sigma_{IJ}\Sigma_{JJ}^{-1}\Sigma_{JI}.$$

- Conditional independence means that

$$\Sigma_{II} - \Sigma_{IJ}\Sigma_{JJ}^{-1}\Sigma_{JI}$$

is diagonal, i.e., that X_i and X_j are conditionally independent for $I = (i, j)$.

MaxDet and the Dual Covariance Selection

The expression for $\Sigma_{I|J}$ is identical to the inverse of the Schur complement of Σ_{JJ} in Σ:

$$
\begin{aligned}
(\Sigma^{-1})_{II} &= \begin{bmatrix} \Sigma_{II} & \Sigma_{IJ} \\ \Sigma_{JI} & \Sigma_{JJ} \end{bmatrix}^{-1}_{II} \\
&= (\Sigma_{II} - \Sigma_{IJ}\Sigma_{JJ}^{-1}\Sigma_{JI})^{-1}.
\end{aligned}
$$

Therefore we require this block to be diagonal or $(\Sigma^{-1})_{ij} = 0$ for $i, j \in I$ with $i \neq j$.

Another way to see that a determinant-maximizing completion of MaxDet must have zeros in the inverse corresponding to the free elements of Σ is by a perturbation argument.

We need the following lemma [Chan, 84].

Lemma

For $v, w, x, y \in \mathbb{R}^n$*,*

$$\det(I + vx^T + wy^T) = (1 + v^T x)(1 + w^T y) - (v^T y)(w^T x).$$

Using the lemma, we consider how the determinant of a symmetric positive definite matrix $A \in \mathbb{R}^{n \times n}$ changes when we perturb a_{ij} (and a_{ji}, by symmetry). Let

$$A(\epsilon) = A + \epsilon(e_i e_j^T + e_j e_i^T),$$

where e_i is the ith column of the identity matrix.

Let $B = A^{-1}$ and partition $B = [b_1, \ldots, b_n]$. Apply the lemma:

$$\begin{aligned}
\det A(\epsilon) &= \det(A(I + \epsilon(b_i e_j^T + b_j e_i^T))) \\
&= \det(A) \det(I + \epsilon(b_i e_j^T + b_j e_i^T)) \\
&= \det(A) \big[(1 + \epsilon b_i^T e_j)(1 + \epsilon b_j^T e_i) - \epsilon^2 (b_i^T e_i)(b_j^T e_j)\big] \\
&= \det(A) \big[(1 + \epsilon b_{ji})(1 + \epsilon b_{ij}) - \epsilon^2 b_{ii} b_{jj}\big] \\
&= \det(A) \big(1 + 2\epsilon b_{ij} + \epsilon^2 (b_{ij}^2 - b_{ii} b_{jj})\big).
\end{aligned}$$

We want to know when $\det A(0)$ is maximal. Since

$$\frac{d}{d\epsilon} \det A(\epsilon)|_{\epsilon=0} = 2 \det(A) b_{ij},$$

we need $b_{ij} = 0$ for a stationary point at $\epsilon = 0$, and from

$$\frac{d^2}{d\epsilon^2} \det A(\epsilon)|_{\epsilon=0} = 2 \det(A)(b_{ij}^2 - b_{ii} b_{jj}) < 0$$

(since B is positive definite), we see that when $b_{ij} = 0$, the quadratic function $\det A()$ has a maximum at $= 0$.

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MaxDet and the Dual Covariance Selection

In general, solving the MaxDet completion problem (or, equivalently, the covariance selection problem) requires solving a convex optimization problem on the set of positive definite matrices [Dahl et al, 08].

- **We develop explicit, easily implementable solutions for some practically important cases arising in the insurance application.**
  - Such solutions are helpful for practitioners and also useful for testing algorithms that tackle the most general problem.

Let Σ denote the solution of the MaxDet completion problem for the partially-specified correlation matrix $\bar{\Sigma}$.

We give a result for an L-shaped pattern of unspecified entries in $\bar{\Sigma}$.

Note that we do not require a unit diagonal, so it applies more generally than just to correlation matrices.

MaxDet and the Dual Covariance Selection

Our solution relies on the results of [Johnson et al., 84].

A positive definite completion necessarily exists for a partially-specified Hermitian matrix if:

- the diagonal entries are specified,
- specified principal minors are positive, and
- the undirected graph of the specified entries is chordal.

Additionally, if a positive definite completion exists, there is a unique matrix, in the class of all positive definite completions, whose determinant is maximal.

MaxDet and the Dual Covariance Selection

Theorem

Consider the symmetric partially specified matrix

$$\bar{\Sigma} = \begin{array}{c c} & \begin{array}{cccc} n_1 & n_2 & n_3 & n_4 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \\ n_3 \\ n_4 \end{array} & \begin{bmatrix} A_{11} & B & C & D \\ B^T & A_{22} & E & F \\ C^T & E^T & A_{33} & G \\ D^T & F^T & G^T & A_{44} \end{bmatrix} \end{array} \in \mathbb{R}^{(n_1+n_2+n_3+n_4)\times(n_1+n_2+n_3+n_4)},$$

where C, E, and F are unspecified, the diagonal blocks A_{ii}, $i = 1:4$ are all positive definite, and all specified principal minors are positive.

The maximal determinant completion is

$$C = DA_{44}^{-1}G^T, \quad F = B^TA_{11}^{-1}D, \quad E = FA_{44}^{-1}G^T.$$

Brief comments on Proof:

- The result can be derived by permuting $\bar{\Sigma}$ so that the unspecified matrices appear in the block (1,3), (1,4), and (2,4) positions and then applying the results of [Dym et al, 81] on completion of block banded matrices.
- The result can also be obtained from [Johnson and Lundquist, 93], in which the unspecified elements of the MaxDet completion are given elementwise in terms of the clique paths in the graph of the specified elements.

Alternatively, we develop an elementary proof based on Gaussian elimination, using the property that Σ^{-1} will contain zeros in the positions of the unspecified entries in $\bar{\Sigma}$.

Basic Proof Steps

- It is easy to check that the graph of the specified entries is block chordal, and therefore a unique determinant maximizing positive definite completion exists!

To find it, we need to solve the linear system

$$\begin{bmatrix} A_{11} & B & C & D \\ B^T & A_{22} & E & F \\ C^T & E^T & A_{33} & G \\ D^T & F^T & G^T & A_{44} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \end{bmatrix},$$

that is,

$$A_{11}X_1 + BX_2 + CX_3 + DX_4 = \Gamma_1, \tag{1}$$

$$B^T X_1 + A_{22}X_2 + EX_3 + FX_4 = \Gamma_2, \tag{2}$$

$$C^T X_1 + E^T X_2 + A_{33}X_3 + GX_4 = \Gamma_3, \tag{3}$$

$$D^T X_1 + F^T X_2 + G^T X_3 + A_{44}X_4 = \Gamma_4, \tag{4}$$

by Gaussian elimination in order to identify the inverse of the matrix $\bar{\Sigma}$.

In this system we can think of C, E, and F as representing any positive definite completions, so that the coefficient matrix is positive definite.

- **We find the determinant maximizing completions by enforcing zeros in relevant blocks of the inverse.**

The following patterns arise frequently in the working below so we assign them variable names to condense the formulae:

$$\mathcal{B} = B - DA_{44}^{-1}F^T, \qquad \mathcal{K} = \mathcal{E} - \mathcal{B}^T\Delta\mathcal{C},$$

$$\mathcal{C} = C - DA_{44}^{-1}G^T, \qquad \mathcal{M} = A_{44}^{-1} + A_{44}^{-1}D^T\Delta DA_{44}^{-1},$$

$$\mathcal{E} = E - FA_{44}^{-1}G^T, \qquad \Delta = (A_{11} - DA_{44}^{-1}D^T)^{-1},$$

$$\mathcal{F} = F - \mathcal{B}^T\Delta D, \qquad \Phi = (A_{22} - FA_{44}^{-1}F^T - \mathcal{B}^T\Delta\mathcal{B})^{-1},$$

$$\mathcal{G} = G - \mathcal{C}^T\Delta D, \qquad \Xi = (A_{33} - GA_{44}^{-1}G^T - \mathcal{C}^T\Delta\mathcal{C} - \mathcal{K}^T\Phi\mathcal{K})^{-1}.$$

- *Inverses in definitions of Δ, Φ, and Ξ exist since matrices being inverted are Schur complements arising in block elimination of the positive definite matrix $\bar{\Sigma}$, so are themselves positive definite.*

We first solve for X_4 in (4), to obtain

$$X_4 = A_{44}^{-1}(\Gamma_4 - D^T X_1 - F^T X_2 - G^T X_3),$$

and substitute this expression into (1) to obtain

$$A_{11}X_1 + BX_2 + CX_3 + DA_{44}^{-1}(\Gamma_4 - D^T X_1 - F^T X_2 - G^T X_3) = \Gamma_1.$$

We can then express X_1 and X_4 in terms of X_2 and X_3 only:

$$\begin{aligned} X_1 &= (A_{11} - DA_{44}^{-1}D^T)^{-1}\Big(\Gamma_1 - DA_{44}^{-1}\Gamma_4 - (B - DA_{44}^{-1}F^T)X_2 - (C - DA_{44}^{-1}G^T)X_3\Big) \\ &= \Delta(\Gamma_1 - DA_{44}^{-1}\Gamma_4 - \mathcal{B}X_2 - \mathcal{C}X_3) \qquad (5) \end{aligned}$$

and

$$\begin{aligned} X_4 &= A_{44}^{-1}\Big(\Gamma_4 - D^T\Delta(\Gamma_1 - DA_{44}^{-1}\Gamma_4 - \mathcal{B}X_2 - \mathcal{C}X_3) - F^T X_2 - G^T X_3\Big) \\ &= A_{44}^{-1}\Big(- D^T\Delta\Gamma_1 + \Gamma_4 + D^T\Delta DA_{44}^{-1}\Gamma_4 - (F^T - D^T\Delta\mathcal{B})X_2 - (G^T - D^T\Delta\mathcal{C})X_3\Big) \\ &= A_{44}^{-1}\Big(- D^T\Delta\Gamma_1 + \Gamma_4 + D^T\Delta DA_{44}^{-1}\Gamma_4 - \mathcal{F}^T X_2 - \mathcal{G}^T X_3\Big). \qquad (6) \end{aligned}$$

Working with (2) next, and separating the X_2 and X_3 variables, we have:

$$\begin{aligned}
A_{22}X_2 &= \Gamma_2 - B^T X_1 - EX_3 - FX_4 \\
&= \Gamma_2 - B^T \Delta(\Gamma_1 - DA_{44}^{-1}\Gamma_4 - \mathcal{B}X_2 - \mathcal{C}X_3) - EX_3 \\
&\quad - FA_{44}^{-1}(-D^T\Delta\Gamma_1 + \Gamma_4 + D^T\Delta DA_{44}^{-1}\Gamma_4 - \mathcal{F}^T X_2 - \mathcal{G}^T X_3) \\
&= -(B^T - FA_{44}^{-1}D^T)\Delta\Gamma_1 + \Gamma_2 - (F - \mathcal{B}^T\Delta D)A_{44}^{-1}\Gamma_4 \\
&\quad + (B^T\Delta\mathcal{B} + FA_{44}^{-1}\mathcal{F}^T)X_2 - \big(E - FA_{44}^{-1}G^T - (B^T - FA_{44}^{-1}D^T)\Delta\mathcal{C}\big)X_3 \\
&= -\mathcal{B}^T\Delta\Gamma_1 + \Gamma_2 - \mathcal{F}A_{44}^{-1}\Gamma_4 + (B^T\Delta\mathcal{B} + FA_{44}^{-1}\mathcal{F}^T)X_2 - (\mathcal{E} - \mathcal{B}^T\Delta\mathcal{C})X_3.
\end{aligned}$$

Therefore

$$(A_{22} - B^T\Delta\mathcal{B} - FA_{44}^{-1}\mathcal{F}^T)X_2 = -\mathcal{B}^T\Delta\Gamma_1 + \Gamma_2 - \mathcal{F}A_{44}^{-1}\Gamma_4 - \mathcal{K}X_3.$$

Notice that the left-hand side simplifies to one of our inverse equations:

$$(A_{22} - B^T\Delta\mathcal{B} - FA_{44}^{-1}\mathcal{F}^T)X_2 = (A_{22} - FA_{44}^{-1}F^T - \mathcal{B}^T\Delta\mathcal{B})X_2 = \Phi^{-1}X_2,$$

$$X_2 = \Phi\big(- \mathcal{B}^T \Delta \Gamma_1 + \Gamma_2 - \mathcal{F} A_{44}^{-1} \Gamma_4 - \mathcal{K} X_3 \big). \tag{7}$$

Substituting (7) into the expressions (5) and (6) we have

$$\begin{aligned} X_1 &= (\Delta + \Delta \mathcal{B} \Phi \mathcal{B}^T \Delta)\Gamma_1 - \Delta \mathcal{B} \Phi \Gamma_2 - \Delta(D - \mathcal{B}\Phi\mathcal{F}) A_{44}^{-1} \Gamma_4 - \Delta(\mathcal{C} - \mathcal{B}\Phi\mathcal{K}) X_3, \\ X_4 &= A_{44}^{-1}(-D^T + \mathcal{F}^T \Phi \mathcal{B}^T)\Delta\Gamma_1 - A_{44}^{-1}\mathcal{F}^T\Phi\Gamma_2 + (\mathcal{M} + A_{44}^{-1}\mathcal{F}^T\Phi\mathcal{F}A_{44}^{-1})\Gamma_4 \\ &\quad + A_{44}^{-1}(\mathcal{F}^T\Phi\mathcal{K} - \mathcal{G}^T) X_3. \end{aligned} \tag{8}$$

Finally, we substitute these expressions into (3) to obtain X_3 in terms of Γ_1, Γ_2, Γ_3, and Γ_4 which is simplified to

$$\begin{aligned} \Xi^{-1} X_3 &= (-\mathcal{C}^T + \mathcal{K}^T \Phi \mathcal{B}^T)\Delta\Gamma_1 - \mathcal{K}^T\Phi\Gamma_2 + \Gamma_3 \\ &\quad + (\mathcal{K}^T\Phi\mathcal{F}A_{44}^{-1} - G\mathcal{M} + C^T \Delta D A_{44}^{-1})\Gamma_4. \end{aligned} \tag{9}$$

Basic Proof Steps

The only blocks of interest in the inverse of $\bar{\Sigma}$ are those that we denote $X_3(\Gamma_1)$ and $X_3(\Gamma_2)$, which are defined by

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} = \begin{bmatrix} A_{11} & B & C & D \\ B^T & A_{22} & E & F \\ C^T & E^T & A_{33} & G \\ D^T & F^T & G^T & A_{44} \end{bmatrix}^{-1} \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ X_3(\Gamma_1) & X_3(\Gamma_2) & \times & \times \\ \times & \times & \times & \times \end{bmatrix} \tag{10}$$

where "$\times$" denotes a block that is not of interest.

Comparing (9) and (10), we find that

$$X_3(\Gamma_2) = -\Xi \mathcal{K}^T \Phi,$$

and we require this expression to be zero for the maximal determinant completion.

- Since Φ and Ξ are inverses, they cannot be zero, therefore we require $\mathcal{K}^T = 0$.

Similarly, we have

$$X_3(\Gamma_1) = \Xi(-\mathcal{C}^T + \mathcal{K}^T \Phi \mathcal{B}^T)\Delta,$$

and since $\mathcal{K}^T = 0$ (and Δ and Ξ are nonsingular) we require that $\mathcal{C} = 0$, which implies that

$$C = DA_{44}^{-1} G^T. \tag{11}$$

The equations $\mathcal{C} = 0$ and $\mathcal{K} = 0$ imply $\mathcal{E} = 0$, and hence

$$E = FA_{44}^{-1} G^T.$$

Denoting by Π the permutation matrix that reverses the order of the blocks in $\bar{\Sigma}$, we have

$$\Pi^T \bar{\Sigma} \Pi = \begin{bmatrix} A_{44} & G^T & F^T & D^T \\ G & A_{33} & E^T & C^T \\ F & E & A_{22} & B^T \\ D & C & B & A_{11} \end{bmatrix}.$$

The block F^T now takes the role of C in the original matrix, so from (11) we obtain, after transposing, $F = B^T A_{11}^{-1} D$.

We have now found the MaxDet completion!

Accuracy and Efficiency of Computation: should be evaluated as follows, avoiding explicit computation of matrix inverses.

Compute Cholesky factorizations:

- $A_{11} = R_{11}^T R_{11}$ and
- $A_{44} = R_{44}^T R_{44}$,

then evaluate unspecified matrix components according to:

$$C = (DR_{44}^{-1})(R_{44}^{-T} G^T), \quad F = (B^T R_{11}^{-1})(R_{11}^{-T} D), \quad E = (FR_{44}^{-1})(R_{44}^{-T} G^T).$$

Each of the terms in parentheses should be evaluated as the solution of a triangular linear system with multiple right handsides.

- The term $R_{44}^{-T} G^T$ can be calculated once and reused.

MaxDet and the Dual Covariance Selection

We identify two useful special cases.

Corollary

Consider the symmetric matrix

$$\begin{array}{cc} & \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} & \begin{bmatrix} A_{11} & B & C \\ B^T & A_{22} & \color{red}{E} \\ C^T & \color{red}{E^T} & A_{33} \end{bmatrix} \end{array} \in \mathbb{R}^{(n_1+n_2+n_3)\times(n_1+n_2+n_3)},$$

where E is unspecified, all the diagonal blocks are positive definite, and all specified principal minors are positive.

The maximal determinant completion is $E = B^T A_{11}^{-1} C$.

Corollary

Consider the symmetric matrix

$$\begin{array}{cc} & \begin{array}{ccc} n_1 & n_2 & n_3 \end{array} \\ \begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array} & \begin{bmatrix} A_{11} & B & C \\ B^T & A_{22} & E \\ C^T & E^T & A_{33} \end{bmatrix} \end{array} \in \mathbb{R}^{(n_1+n_2+n_3)\times(n_1+n_2+n_3)},$$

where C is unspecified, all the diagonal blocks are positive definite, and all specified principal minors are positive.

The maximal determinant completion is $C = BA_{22}^{-1}E$.

Now we consider a pattern of unspecified elements that arises when (for example) an insurance company has four business units where correlations between BU-specific risks are known

- described by the specified blocks A_{11}, A_{22}, A_{33} and A_{44} and
- all the correlations are known for the first group of risks (for example, risk drivers such as interest rates or exchange rates).

So here we have a complete first block row and column, and **this case cannot be obtained by permuting rows and columns in Theorem 7**.

MaxDet and the Dual Covariance Selection

Theorem

Consider the symmetric matrix

$$\begin{array}{c|cccc} & n_1 & n_2 & n_3 & n_4 \\ \hline n_1 & \mathbf{A_{11}} & \mathbf{B} & \mathbf{C} & \mathbf{D} \\ n_2 & \mathbf{B^T} & A_{22} & E & F \\ n_3 & \mathbf{C^T} & E^T & A_{33} & G \\ n_4 & \mathbf{D^T} & F^T & G^T & A_{44} \end{array} \in \mathbb{R}^{(n_1+n_2+n_3+n_4)\times(n_1+n_2+n_3+n_4)},$$

where E, F, and G are unspecified, all the diagonal blocks are positive definite, and all specified principal minors are positive.

The maximal determinant completion of the matrix is

$$E = B^T A_{11}^{-1} C, \quad F = B^T A_{11}^{-1} D, \quad G = C^T A_{11}^{-1} D.$$

Finally, we consider the case where C, E and F are unspecified, and B and G are partly specified.

Theorem

Consider the symmetric matrix

$$\bar{\Sigma} = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & n_3 & n_4 \end{array} \\ \begin{bmatrix} A_{11} & \color{blue}{B} & \color{red}{C} & D \\ \color{blue}{B^T} & A_{22} & \color{red}{E} & \color{red}{F} \\ \color{red}{C^T} & \color{red}{E^T} & A_{33} & \color{blue}{G} \\ D^T & \color{red}{F^T} & \color{blue}{G^T} & A_{44} \end{bmatrix} \end{array} \in \mathbb{R}^{(n_1+n_2+n_3+n_4)\times(n_1+n_2+n_3+n_4)},$$

where C, E, and F are unspecified, B and G are partly specified (possibly fully unspecified), all the diagonal blocks are positive definite, all specified principal minors are positive, and the graph of the specified entries is block chordal.

If B and G are fully unspecified then the maximal determinant completion of the matrix is

$$\Sigma = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \end{array} \begin{array}{c} \begin{array}{cccc} n_1 & n_2 & n_3 & n_4 \end{array} \\ \begin{bmatrix} A_{11} & 0 & 0 & D \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & A_{33} & 0 \\ D^T & 0 & 0 & A_{44} \end{bmatrix} \end{array}. \tag{12}$$

Otherwise, the maximal determinant completion of B and G is independent of the entries in D.

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MaxDet for Large Block Structures

In general we can extend these results in a recursive iterative manner to get results on larger matrices.

In some examples we need additional results for larger block structures

- Eg. applications with many business units with many BU-specific risks.

Correlations are assumed to be known between all “standard” risk drivers,

- typically the market risks in all business units.

This is because there is generally sufficient data to calculate correlations between equity indices, interest rates, and credit spreads, say, across economies.

MaxDet for Large Block Structures

Table: Example of partial internal model Integration Technique, where one of the constituents of the standard formula (SF) market risk module (currency risk) has been included in the IM
$\Rightarrow$ so correlations are required between the SF market risk submodules and the other SF modules (that is, the green starred cells).

Module	**Sub-module**											
Internal Model		*1*	0.25	0.6	0.55	0.65	0	0.4	0.6	0.2	0.3
SF Market risk	Interest rate	*0.25*	1	0	0	0	0	*	*	*	*
	Equity	*0.6*	0	1	0.75	0.75	0	*	*	*	*
	Property	*0.55*	0	0.75	1	0.5	0	*	*	*	*
	Spread	*0.65*	0	0.75	0.5	1	0	*	*	*	*
	Concentration	*0*	0	0	0	0	1	*	*	*	*
SF Default		*0.4*	*	*	*	*	*	1	0.25	0.25	0.5
SF Life		*0.6*	*	*	*	*	*	0.25	1	0.25	0
SF Health		*0.2*	*	*	*	*	*	0.25	0.25	1	0
SF Non-Life		*0.3*	*	*	*	*	*	0.5	0	0	1

MaxDet for Large Block Structures

The extension relies on the observation that if the B or G blocks in the previous Theorem have unknown entries then the maximal determinant completions for these blocks are independent of the other entries in the matrix.

- The next Theorem shows the calculation for four business units, laid out as two instances of the case in the previous Theorem, in the upper left and bottom right corners of the matrix $\bar{\Sigma}$.
- Three business units can be obtained as a special case where one business unit has empty elements.
- More than four business units can be accommodated by repeated applications of the previous Theorem.

Theorem

Consider the symmetric matrix

$$
\bar{\Sigma} = \begin{array}{c} \\ n_1 \\ n_2 \\ n_3 \\ n_4 \\ 12pt\,n_5 \\ n_6 \\ n_7 \\ n_8 \end{array}
\begin{array}{c}
\begin{array}{cccccccc} n_1 & n_2 & n_3 & n_4 & n_5 & n_6 & n_7 & n_8 \end{array} \\
\left[\begin{array}{cccc|cccc}
A_{11} & A_{12} & {\color{red}A_{13}} & A_{14} & A_{15} & {\color{red}A_{16}} & {\color{red}A_{17}} & A_{18} \\
A_{12}^T & A_{22} & {\color{red}A_{23}} & {\color{red}A_{24}} & {\color{red}A_{25}} & {\color{red}A_{26}} & {\color{red}A_{27}} & {\color{red}A_{28}} \\
{\color{red}A_{13}^T} & {\color{red}A_{23}^T} & A_{33} & A_{34} & {\color{red}A_{35}} & {\color{red}A_{36}} & {\color{red}A_{37}} & {\color{red}A_{38}} \\
A_{14}^T & {\color{red}A_{24}^T} & A_{34}^T & A_{44} & A_{45} & {\color{red}A_{46}} & {\color{red}A_{47}} & A_{48} \\
\hline
A_{15}^T & {\color{red}A_{25}^T} & {\color{red}A_{35}^T} & A_{45}^T & A_{55} & A_{56} & {\color{red}A_{57}} & A_{58} \\
{\color{red}A_{16}^T} & {\color{red}A_{26}^T} & {\color{red}A_{36}^T} & {\color{red}A_{46}^T} & A_{56}^T & A_{66} & {\color{red}A_{67}} & {\color{red}A_{68}} \\
{\color{red}A_{17}^T} & {\color{red}A_{27}^T} & {\color{red}A_{37}^T} & {\color{red}A_{47}^T} & {\color{red}A_{57}^T} & {\color{red}A_{67}^T} & A_{77} & A_{78} \\
A_{18}^T & {\color{red}A_{28}^T} & {\color{red}A_{38}^T} & A_{48}^T & A_{58}^T & {\color{red}A_{68}^T} & A_{78}^T & A_{88}
\end{array}\right]
\end{array}
$$

$$
= \begin{array}{c}
\begin{array}{cc} n_1+n_2+n_3+n_4 & n_5+n_6+n_7+n_8 \end{array} \\
\left[\begin{array}{cc} N & Q \\ Q^T & M \end{array}\right]
\end{array},
$$

where the diagonal blocks A_{ii} are all positive definite, the specified principal minors are all positive, and the red blocks are unspecified.

Theorem (Continued)

The maximal determinant completion of the matrix is

$$A_{13} = A_{14}A_{44}^{-1}A_{34}^T, \qquad A_{24} = A_{12}^T A_{11}^{-1} A_{14}, \qquad A_{23} = A_{24}A_{44}^{-1}A_{34}^T,$$
$$A_{57} = A_{58}A_{88}^{-1}A_{78}^T, \qquad A_{68} = A_{56}^T A_{55}^{-1} A_{58}, \qquad A_{67} = A_{68}A_{88}^{-1}A_{78}^T,$$
$$C = DH^{-1}G^T, \qquad F = B^T A^{-1} D, \qquad E = FH^{-1}G^T,$$

where

$$A = \begin{bmatrix} A_{11} & A_{14} \\ A_{14}^T & A_{44} \end{bmatrix}, \qquad B = \begin{bmatrix} A_{12} & A_{13} \\ A_{42} & A_{43} \end{bmatrix},$$
$$C = \begin{bmatrix} A_{16} & A_{17} \\ A_{46} & A_{47} \end{bmatrix}, \qquad D = \begin{bmatrix} A_{15} & A_{18} \\ A_{45} & A_{48} \end{bmatrix},$$
$$E = \begin{bmatrix} A_{26} & A_{27} \\ A_{36} & A_{37} \end{bmatrix}, \qquad F = \begin{bmatrix} A_{25} & A_{28} \\ A_{35} & A_{38} \end{bmatrix},$$
$$G = \begin{bmatrix} A_{65} & A_{68} \\ A_{75} & A_{78} \end{bmatrix}, \qquad H = \begin{bmatrix} A_{55} & A_{58} \\ A_{58}^T & A_{88} \end{bmatrix}.$$

Conclusions

- We have derived explicit solutions for completions with maximal determinant of a wide class of partially specified correlation matrices that arise in the context of insurers calculating economic capital requirements.
- The patterns supported are block diagonal, with either cross-shaped or (inverted) L-shaped gaps on the off-diagonal.
- The solutions are easy to evaluate, being expressed in terms of products and inverses of known matrices.
- Possible directions for future work include developing explicit solutions for more general patterns of unspecified entries and allowing semidefinite diagonal blocks and zero principal minors.