Modelling optimal decisions in retirement under expected utility stochastic control framework

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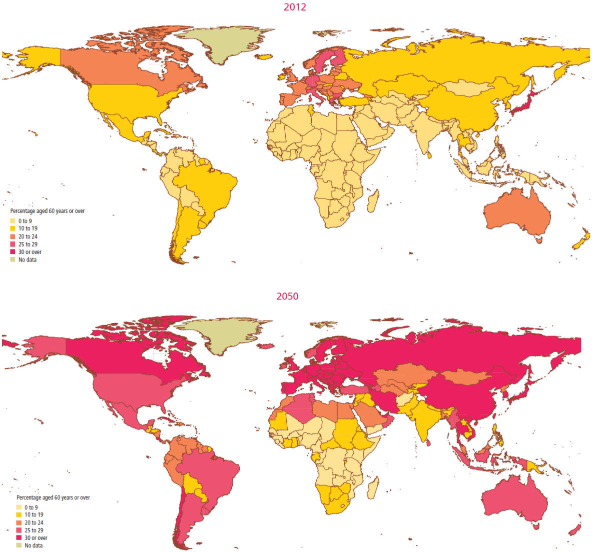
Population Aging - United Nations (2013): percentage aged 60 years or over, 2012 vs 2050 forecast



Motivation

- Australia's accumulation benefit pension system is still young, but superannuation assets already accumulated $2.259tn (ASFA, 2017).
- More retirees - due to both increased life expectancy and an ageing population. Currently 15% of population is 65+ people.
- Age and Service Pension payments will change from 2.9% of GDP in 2015 to 3.6% in 2055 (the number of 65+ will more than double).
- 2017-18 Australian government budget – social security and welfare is 38% of taxpayers money (assistance for aged Australians is the largest part).
- Limited knowledge amongst retirees (and advisors) to manage funds and Age Pension (Spicer et al., 2013). Retirement funds have changed from monthly benefit to lump sum at retirement.
- **Modelling consumption, bequest, home ownership, and investment is important for retirees and the Australian Pension system. Current modelling typically is scenario based and ignores the dynamic response or too simplistic.**

Australian superannuation

- Three pillars - **superannuation guarantee, private savings and government provided Age Pension**.
- Superannuation guarantee contribution rate is 9.5%, set to increase 2021-2022 and reach to 12% by 2025-2026.
- Means-tested Age Pension: subject to income-test and asset-test, and entitlement age of 65.5.
- **Family home is excepted from Age pension asset-test.**
- Income-test based on actual income, deemed income and drawdown of allocated pension accounts.
- Policies and regulations are constantly changing.
- **Allocated pension accounts** are purchased with superannuation, and subject to **minimum withdrawal rates**; current rates (2017–):

Age	$\leq$64	65–74	75–79	80–84	85–89	90–94	$\leq$95
Min. drawdown	4%	5%	6%	7%	9%	11%	14%

	PRE2015	POST2015	POST2017
Full Age Pension singles ($P^{\mathrm{S}}_{\mathrm{max}}$)	\$22,721	\$22,721	\$22,721
Full Age Pension couples ($P^{\mathrm{C}}_{\mathrm{max}}$)	\$34,252	\$34,252	\$34,252
Income-Test	Drawdown	Deemed	Deemed
Threshold singles (L^{S}_{I})	\$4264	\$4264	\$4264
Threshold couples (L^{C}_{I})	\$7592	\$7592	\$7592
Rate of reduction (ϖ^{d}_{I})	\$0.5	\$0.5	\$0.5
Deeming threshold singles (κ^{S})	-	\$49,200	\$49,200
Deeming threshold couples (κ^{C})	-	\$81,600	\$81,600
Deeming rate below κ^{d} (ς_{-})	-	1.75%	1.75%
Deeming rate above κ^{d} (ς_{+})	-	3.25%	3.25%
Asset-Test			
Threshold homeowners singles ($L^{S,h=1}_{\mathrm{A}}$)	\$209,000	\$209,000	\$250,000
Threshold homeowners couples ($L^{C,h=1}_{\mathrm{A}}$)	\$296,500	\$296,500	\$375,000
Threshold non-homeowners singles ($L^{S,h=0}_{\mathrm{A}}$)	\$360,500	\$360,500	\$450,000
Threshold non-homeowners couples ($L^{C,h=0}_{\mathrm{A}}$)	\$448,000	\$448,000	\$575,000
Rate of reduction (ϖ^{d}_{A})	\$0.039	\$0.039	\$0.078

Expected Utility Model: Assumptions

- Agent (household) is an expected utility maximiser based on **hyperbolic absolute risk aversion (HARA)**.
- Assuming time-separable additive utility functions for consumption, housing and bequest. Each year alive the retiree receives utility from consumption and housing, and bequest in case of death.
- Start at retirement $t = t_0$ where the retiree can allocate wealth into **housing and an allocated pension account**. Lives no longer than terminal time T.
- Starts as either a couple or single. Couples have mortality risk where if one spouse dies and it becomes a single household.
- **All wealth is held in an allocated pension account, which does not attract taxes on capital gains.**
- Each period the retiree receives Age Pension, consumes part of his wealth and allocates the remaining into a risky asset and risk-free asset.

Model Definition

Denote a state vector as $X_t = (W_t, G_t, H) \in \mathcal{W} \times \mathcal{G} \times \mathcal{H}$ where $W_t \in \mathcal{W} = \mathbb{R}^+$ denotes the current level of wealth, $G_t \in \mathcal{G} = \{\Delta, 0, 1, 2\}$ whether the agent is dead, died this period, alive in a single household or alive in a couple household. $H \in \mathcal{H} = \mathbb{R}^+$ denotes wealth invested in housing at t_0. Realisations of W_t and G_t are w_t and g_t respectively.

The utility received at times t is subject to the control variables

- α_t - proportion drawdown of liquid wealth,
- δ_t - proportion liquid wealth allocated to risky assets,

and the decision variable

- ϱ - wealth allocated to housing only at time $t = t_0$.

Define a decision rule $\pi_t(x_t) = (\alpha_t, \delta_t)$ which is the action at time t and depends on the current state x_t. Then a sequence (policy) of decision rules is given by $\pi = (\pi_{t_0}, \pi_{t_0+1}, ..., \pi_{T-1})$ for $t = t_0, t_0 + 1, ..., T - 1$.

Model Definition

Wealth process is driven by stochastic return $Z_{t+1} \overset{i.i.d}{\sim} \mathcal{N}(\mu, \sigma)$ and deterministic risk-free rate r, and controlled by drawdown α_t and risky asset allocation δ_t, given by

$$W_{t+1} = (W_t - \alpha_t W_t)\left(\delta_t e^{Z_t} + (1 - \delta_t)e^{r_t}\right), \tag{1}$$

$$\text{s.t} \quad C_t = \alpha_t W_t + P_t,$$
$$W_t \geq 0,$$
$$W_{t_0} = \mathrm{W} - H,$$
$$H \in \{0, [H_L, \mathrm{W}]\},$$

where W_t is the liquid wealth before withdrawal, r_t is the time dependent but deterministic real risk-free rate and W is the initial total wealth.

Model Definition

Each period the agent receives utility, given by

$$R_t(W_t, G_t, \alpha_t, H) = \begin{cases} U_C(C_t, G_t, t) + U_H(H, G_t), & \text{if } G_t = 1, 2, \\ U_B(W_t, H), & \text{if } G_t = 0, \\ 0 & \text{if } G_t = \Delta, \end{cases} \tag{2}$$

with terminal condition $(t = T)$ given by

$$\widetilde{R}(W_T, G_T, H) = \begin{cases} U_B(W_T, H), & \text{if } G_T \geq 0 \\ 0, & \text{if } G_T = \Delta. \end{cases} \tag{3}$$

We need to find a solution of the following problem

$$\widetilde{V} := \max_{\varrho} \left[\sup_{\pi} \mathbb{E}^{\pi}_{t_0} \left[\beta_{t_0,T} \widetilde{R}(W_T, G_T, H) + \sum_{t=t_0}^{T-1} \beta_{t_0,t} R_t(W_t, G_t, \alpha_t, H) \right] \right] \tag{4}$$

where $\mathbb{E}^{\pi}_{t_0}[\cdot]$ is the expectation conditional on information and decision at time $t = t_0$ and $\beta_{t,t'}$ is the discounting from t to t'.

Model Definition

Consumption is based on drawdown of wealth and Age Pension received. Utility is received from consumption exceeding the consumption floor.

$$U_C(C_t, G_t, t) = \frac{1}{\psi^{t-t_0}\gamma_d}\left(\frac{C_t - \bar{c}_d}{\zeta_d}\right)^{\gamma_d}, d = \begin{cases} \mathrm{C}, & \text{if } G_t = 2 \quad \text{(couple)}, \\ \mathrm{S}, & \text{if } G_t = 1 \quad \text{(single)}, \end{cases} \tag{5}$$

where $\gamma_d \in (-\infty, 0)$ is the risk aversion, $\bar{c}_d$ the consumption floor, C_t the consumption for year t and ζ_d the scaling factor to normalise between singles and couples. Let $\psi \in [1, \infty)$ be the utility parameter for the health proxy.

Model Definition

Bequest utility function is defined as

$$U_B(W_t, H) = \left(\frac{\theta}{1-\theta}\right)^{1-\gamma_{\mathrm{S}}} \frac{\left(\frac{\theta}{1-\theta}a + W_t + H\right)^{\gamma_{\mathrm{S}}}}{\gamma_{\mathrm{S}}}, \quad (6)$$

where W_t is the liquid wealth available for bequest, γ_{S} the risk aversion of bequest utility (same as consumption risk aversion for singles), $a \in R^+$ is the threshold for luxury bequest and $\theta \in [0, 1)$ is the degree of altruism.

Model Definition

Housing generates utility through a flow of services, approximated with the house value,

$$U_H(H) = \frac{1}{\gamma_{\mathrm{H}}} \left(\frac{\lambda_d H}{\zeta_d} \right)^{\gamma_{\mathrm{H}}}, \qquad (7)$$

where γ_H is the risk aversion parameter for housing (allowed to be different from risk aversion for consumption and bequest), ζ_d is the same scaling factor as for consumption, $H \in R^+$ is the market value of the family home at time of purchase and $\lambda_d \in [0, 1]$ is the preference of housing defined as a proportion of the market value.

Age Pension Formula

Over 90% of income comes from allocated pensions, hence wealth in the asset test equals allocated pension and the drawdown is considered income. The means test is subject to different thresholds for single, couples and whether they are homeowners or not, where

- P^d_{max} is the full Age Pension.
- L^d is the threshold for the asset/income test.
- ϖ^d is the taper rate for assets/income test.
- $h = \{0, 1\}$ whether the retiree is a homeowner or not.

Asset test: $P_{\mathrm{A}}(W_t) = P^d_{\max} - (W_t - L^d_{\mathrm{A}})\varpi^d_{\mathrm{A}}$.

Income test: $P_{\mathrm{I}}(\alpha_t W_t, t) = P^d_{\max} - (\alpha_t W_t - M(t) - L^{d,h}_{\mathrm{I}})\varpi^d_{\mathrm{I}}$.

Income test deduction: $M(t) = \frac{W_{t_0}}{e_{t_0}}(1 + \widetilde{r})^{t_0 - t}$, where e_{t_0} is the life expected at age t_0 and $\widetilde{r}$ the inflation.

Combined Age Pension formula

$$P_t := f(\alpha_t, W_t, t) = \max\left[0, \min\left[P^d_{\max}, \min\left[P_{\mathrm{A}}(W_t), P_{\mathrm{I}}(\alpha_t W_t, t)\right]\right]\right]. \tag{8}$$

New Age Pension policy

The calibrated model is already outdated:

- From 2015, deemed income is used in allocated pension (previously drawdown).
- In 2017, the asset-test thresholds were 'rebalanced' and taper rate doubled.

Deemed income in the pension function:

$$P_{\mathrm{I}} := P^{d}_{\max} - \left(P_{\mathrm{D}}(W_t) - L^{d}_{\mathrm{I}}\right) \varpi^{d}_{\mathrm{I}}, \tag{9}$$

$$P_{\mathrm{D}}(W_t) = \varsigma_{-} \min\left[W_t, \kappa^{d}\right] + \varsigma_{+} \max\left[0, W_t - \kappa^{d}\right]. \tag{10}$$

Stochastic control problem formulation

- Denote an action space of $(\alpha_t, \delta_t, \varrho) \in \mathcal{A} = (-\infty, 1] \times [0, 1] \times \{0, [\frac{H_L}{W}, 1]\}$ for $t = t_0$, and $(\alpha_t, \delta_t) \in \mathcal{A} = (-\infty, 1] \times [0, 1]$ for $t = t_0 + 1, ..., T - 1$.
- Denote an admissible space of state-action combination as $D_t(x_t) = \{\pi_t(x_t) \in A \mid \alpha_t \geq \frac{\bar{c}_d - P_t}{w_t}\}$ which contains the possible actions for the current state.
- The transition function is our wealth process $T_t(W_t, \alpha_t, \delta_t, z_{t+1}) := W_{t+1} = W_t(1 - \alpha_t) \times (\delta_t e^{z_{t+1}} + (1 - \delta_t) e^{r_t})$.
- Denote the stochastic transitional kernel as $Q_t(dx' | x, \pi_t(x))$ which represents the probability to reach a state in $dx' = (dw_{t+1}, g_{t+1})$ at time $t + 1$ if action $\pi_t(x)$ is applied in state x at time t.

Solved numerically with backwards recursion on the Bellman equation, starting from terminal condition

$$V_T(X_T) = \widetilde{R}_T(W_T, G_T, H), \tag{11}$$

and for each $t < T$

$$V_t(X_t) = \sup_{\pi_t(x_t) \in D_t(x_t)} \left\{ R_t(W_t, G_t, \alpha_t, H) + \beta_{t,t+1} \, \mathbb{E}_t^{\pi} \left[V_{t+1}(X_{t+1}) \mid X_t \right] \right\}. \tag{12}$$

Numerical solution

- Discretise wealth state W and house state H on log-equidistant grid and solve recursively with backwards induction.
- Family status state G can be avoided by weighting the reward function with survival probabilities in the value function.
- Numerical integration by Gauss-Hermite Quadrature, with 5 nodes.
- Interpolation via shape preserving Piecewise Cubic Hermite Interpolation Polynomial (PCHIP), which preserves the monotonicity and concavity.
- Decision variable for housing enough to solve at $t = t_0$.

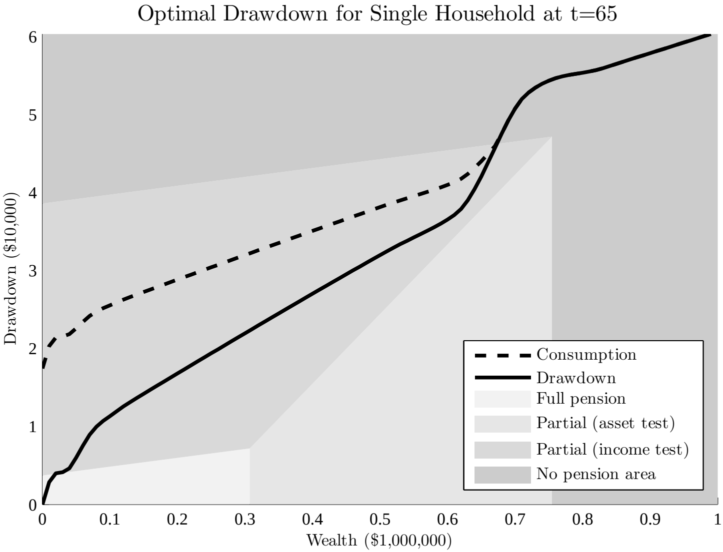
Calibration

- Data were taken from Australian Bureau of Statistics Household Expenditure Survey (HES) 2009-2010, and Survey of Income and Household (SIH) 2009-2010.
- Only a snapshot, does not offer data of cohorts over time.
- Data aggregated based on households in retirement and not part of the work force, split over both single (2,038 data points) and couple households (2,017 data points).
- Calibrate parameters via maximum likelihood estimation on consumption and housing samples.

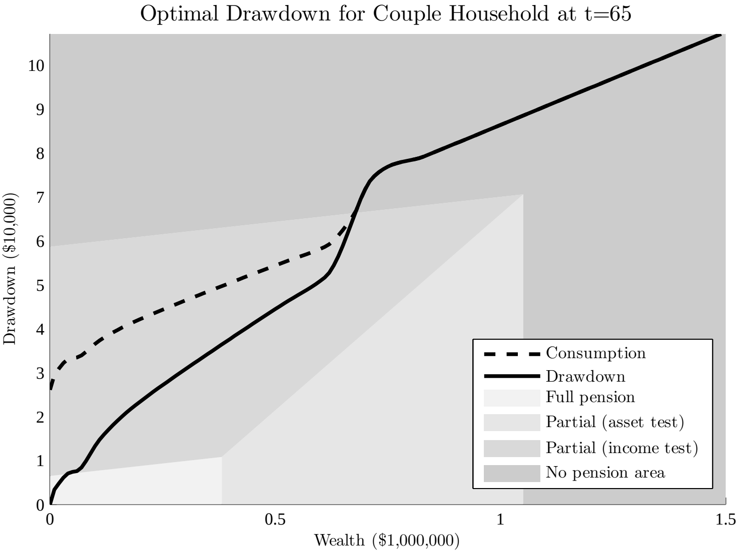
Calibrated utility parameters with standard error:

	γ_S	γ_C	γ_H	θ	a	$\bar{c}_S$	$\bar{c}_C$	ψ	λ
Value	- 2.77	- 2.29	-2.58	0.54	26 741	11 125	18 970	1.47	0.037
Std. Error	0.12	0.14	0.19	0.03	1 377	1 011	1 682	0.04	0.006

Calibration Output - Optimal Consumption



Calibration Output - Optimal Consumption



Calibration Output - Optimal Consumption

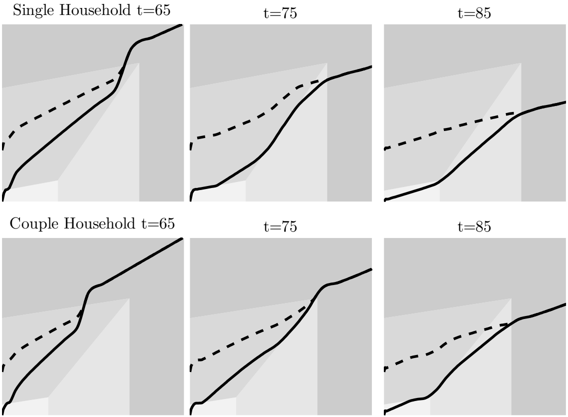


Figure: Comparison of optimal consumption over time.

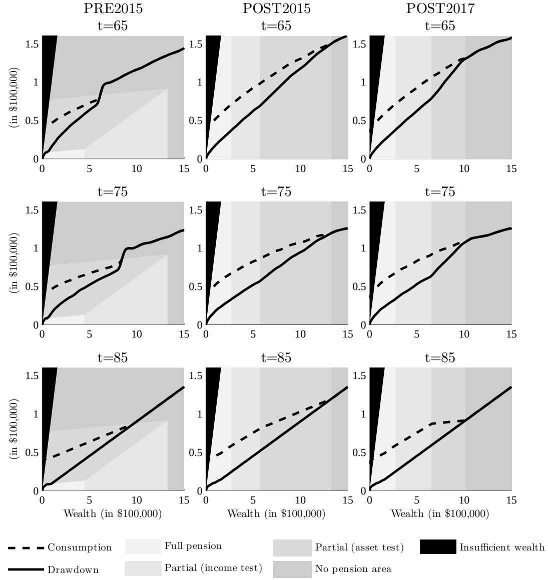


Figure: Optimal drawdown and consumption for non-homeowner couple households for a given liquid wealth at the age t, under the three different policy scenarios in the case of low returns ($\mu = 0.0325$).

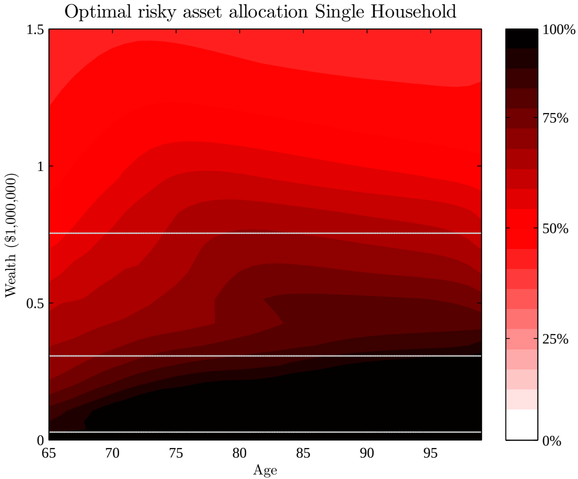


Figure: Optimal allocation of risky assets for single non-homeowners. The horizontal lines (from bottom up) show the threshold a, the threshold for partial Age Pension due to asset test, and the threshold for no Age Pension due to asset test.

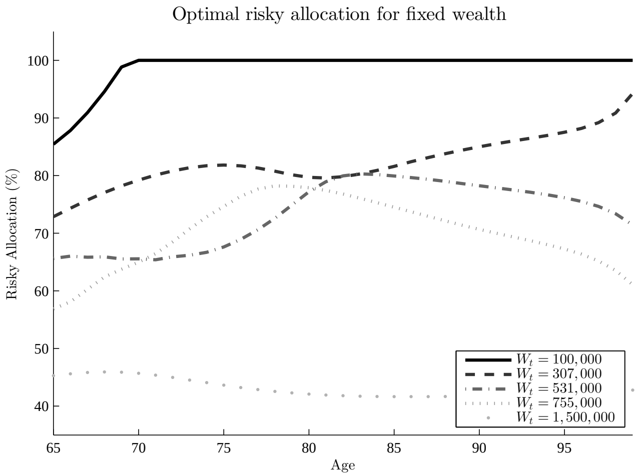


Figure: Optimal allocation of risky assets for single non-homeowners, given fixed wealth. The wealth levels correspond to full pension, the lower threshold of the asset test, partial pension due to asset test, the upper threshold of the asset test, and no pension.

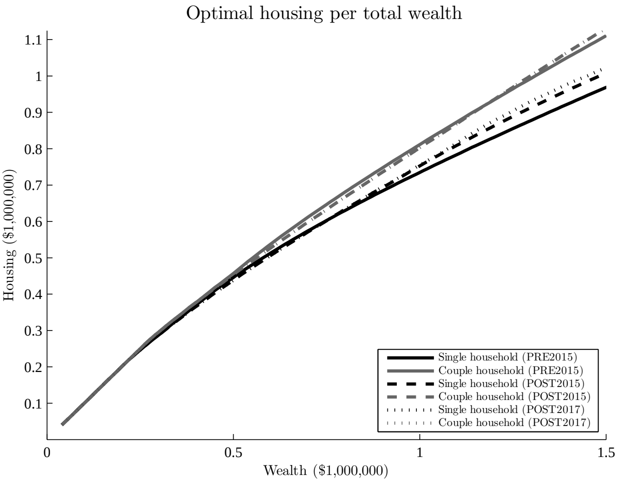


Figure: Optimal housing allocation given by total initial wealth W for single and couple households, under the three policy scenarios with the low return ($\mu = 0.0325$).

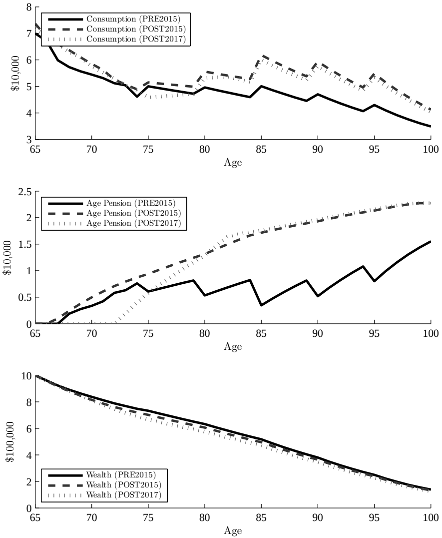
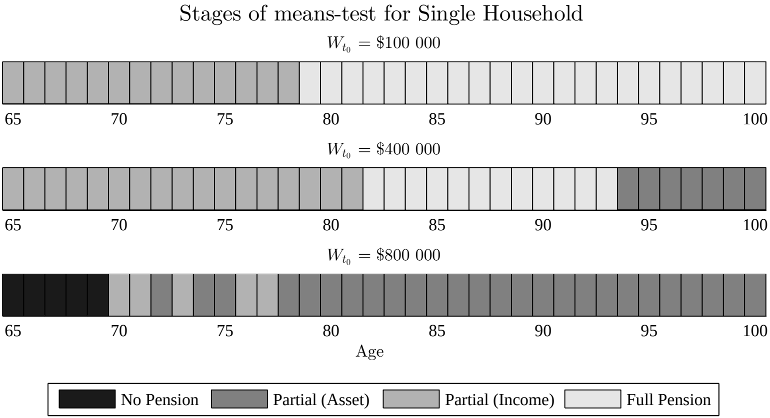


Figure: Comparison of consumption, Age Pension and wealth over a retirees lifetime with the three different policy scenarios. The retiree starts with $1m liquid wealth, which grows with the low expected return each year ($\mu = 0.0325$), and drawdown follows the optimal drawdown paths under each policy.

Calibration Output - Phases of Means-test



Conclusions - Calibrated model

- Optimal drawdown is highly sensitive to the means test early in retirement due to the number of expected years remaining to receive Age Pension, but decreases with time so optimal consumption becomes approximately linear.
- **The Age Pension works as a buffer against investment losses, hence optimal allocation to risky asset increases rapidly when the asset test binds and suggest 100% risky allocation when full Age Pension is received.**
- Optimal housing is similar between single and couple households in terms of proportion of wealth, but house value will differ due to different wealth levels. The high allocation for lower wealth matches the characteristics where households with lower wealth levels tend to have the family home as their only asset.

Least-Squares Monte Carlo method for model extensions

The model should be extended with additional deposit account, stochastic interest rate, housing decisions, reverse mortgage, annuitization, ...

- Additional states and stochastic variables make a quadrature based numerical solution computationally infeasible.
- Least-Squares Monte Carlo (LSMC) is an approximate method for solving stochastic control problems, introduced by Longstaff and Schwartz in 2001 for valuation of American options.
- Essentially a simulation and regression algorithm, where random paths are simulated and the conditional expectation in Bellman equation is approximated with a regression function, then solved via backwards recursion for stochastic control problems.
- Original exogenous model extended in Kharroubi et al. (2014) with endogenous state variables and control randomisation.

Problem Definition

Let $(\Omega,\ \mathcal{F},\ \{\mathcal{F}_t\}_{0\le t\le T},\ \mathbb{P})$ be a filtered complete probability space and $\mathcal{F}_t$ represents the information available before a discrete time t. All the processes introduced below are well defined and adapted to $\{\mathcal{F}_t\}_{t\ge 0}$.

Controlled state variable $X^\pi = (X_t^\pi)_{t=t_0,\dots,T} \in \mathcal{X}$.
Control $\pi = (\pi_t)_{t=t_0,\dots,T} \in \mathcal{A}$.
Random disturbance $Z = (Z_t)_{t=t_0,\dots,T} \in \mathcal{Z}$.
State variable evolution $X_{t+1}^\pi = T(X_t^\pi, \pi_t, Z_{t+1})$.

Maximise the expected value of a value function

$$V(t_0, x) = \sup_\pi \mathbb{E}\left[\beta^{T-t_0} G_T(X_T^\pi) + \sum_{t=t_0}^{T-1} \beta^{t-t_0} R_t(X_t^\pi, \pi_t) | X_{t_0}^\pi = x, \pi_t\right],$$

where G_T and R_t are functions satisfying integrability conditions.

Problem Definition

This type of problem can be solved with backward recursion of the Bellman equation, where

$$\begin{aligned} V(T,x) &= G_T(x), \\ V(t,x) &= \sup_{\pi_t} \left\{ R_t(x,\pi_t) + \mathbb{E}\left[\beta V(t+1, X^{\pi}_{t+1}) | X^{\pi}_t = x; \pi_t\right] \right\}. \end{aligned} \tag{13}$$

Approximate the conditional probability in equation (13)

$$\Phi(X^{\pi}_t, \pi_t) = \mathbb{E}\left[\beta V(t+1), X^{\pi}_{t+1}) | X^{\pi}_t; \pi_t\right], \tag{14}$$

by a regression scheme with independent variables X^{π}_t and π_t.

Arguments for LSMC

Arguments *for* LSMC:

- Does not suffer from "curse of dimensionality", hence faster than other numerical methods as the number of state variables increase.
- No restrictions on dynamics of stochastic processes (contrary to PDE's). Enough to be able to simulate a path.
- Parametric estimate in feedback form of control (no grid required).

Arguments *against* LSMC:

- Approximate method only, and can have substantial errors piling up over multiple periods.
- Can be computationally intensive, especially for the optimisation of control variables.
- Basis function can be difficult to find and is highly problem specific.

LSMC for models with Utility Functions

There are difficulties with LSMC in the case of utility type models; difficult to fit due to extreme curvature over the full sample (extreme heteroskedasticity). Proposed method: regressing on the *transformed* value function and adjusting for the retransformation bias.

Define a transformation H^{-1} such that $H^{-1}(H(x)) = x$. Let $\mathbf{L}(X_t, \pi_t)$ be a vector of basis functions and $\mathbf{\Lambda}_t$ the corresponding regression coefficients vector, such that

$$\mathbb{E}\left[H^{-1}(\beta V_{t+1}(X_{t+1}))|X_t; \pi_t\right] = \mathbf{\Lambda}_t' \mathbf{L}(X_t, \pi_t). \quad (15)$$

If M independent Markovian paths of state and control variables are simulated, one can consider the ordinary linear regression

$$\begin{aligned} &H^{-1}(\beta V_{t+1}(X_{t+1}^m)) = \mathbf{\Lambda}_t' \mathbf{L}(X_t^m, \pi_t^m) + \epsilon_t^m, \\ &\epsilon_t^m \overset{iid}{\sim} F_t(\cdot), \ \ \mathbb{E}[\epsilon_t^m] = 0, \ \ \mathrm{var}[\epsilon_t^m] = \sigma_t^2, \ \ m = 1, ..., M \end{aligned} \quad (16)$$

$$\hat{\mathbf{\Lambda}}_t = \arg\min_{\mathbf{\Lambda}} \sum_m \left[H^{-1}(V(t, X_t^m)) - \mathbf{\Lambda}' L(X_t^m, \pi_t^m)\right]^2. \quad (17)$$

Duan's Smearing Estimate (Duan, 1983)

Our objective is to estimate $\Phi_t(X_t, \pi_t) = \mathbb{E}\left[\beta V_{t+1}(X_{t+1}) | X_t; \pi_t\right]$:

$$\Phi_t(X_t, \pi_t) := H^B(\mathbf{\Lambda}_t' \mathbf{L}(X_t, \pi_t)) = \int H(\mathbf{\Lambda}_t' \mathbf{L}(X_t, \pi_t) + \epsilon_t) dF_t(\epsilon_t), \quad (18)$$

where $F_t(\epsilon_t)$ is the distribution of disturbance term ϵ_t. Obviously,

$$\widehat{H}^B(\widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t, \pi_t)) = H(\widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t, \pi_t)) \quad (19)$$

will be neither unbiased nor consistent unless the transformation is linear. If a specific distribution is assumed for ϵ_t, then the integration in can be performed. Otherwise, the empirical distribution of residuals

$$\widehat{\epsilon}_t^m = H^{-1}(\beta V_{t+1}(X_{t+1}^m)) - \widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t^m, \pi_t^m), \quad (20)$$

can be used to perform the required integration leading to the following **Smearing Estimate**:

$$\widehat{H}^B(\widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t, \pi_t)) = \frac{1}{M} \sum_{m=1}^{M} H(\widehat{\mathbf{\Lambda}}_t' \mathbf{L}(X_t, \pi_t) + \widehat{\epsilon}_t^m), \quad (21)$$

Controlled Heteroskedasticity

If heteroskedasticity is present in the regression with respect to state and control variables, a method that accounts for the heteroskedasticity is required. In this case the conditional variance can be modelled as

$$\mathrm{var}[\epsilon_t | X_t, \pi_t] = [\Omega(\boldsymbol{\mathcal{L}}'_t \mathbf{C}(X_t, \pi_t))]^2, \quad (22)$$

where $\Omega(\cdot)$ is some positive function, $\boldsymbol{\mathcal{L}}_t$ is the vector of coefficients and $\mathbf{C}(X_t, \pi_t)$ is a vector of basis functions. There are various standard ways to find estimates $\widehat{\boldsymbol{\mathcal{L}}}_t$, the one we use is based on the linear regression of the log of squared residuals $\widehat{\epsilon}_t^m$. Then, one can use the **Smearing Estimate with Controlled Heteroskedasticity**:

$$\widehat{H}^B(\widehat{\boldsymbol{\Lambda}}'_t \mathbf{L}(X_t, \pi_t)) = \frac{1}{M} \sum_{m=1}^{M} H\left(\widehat{\boldsymbol{\Lambda}}'_t \mathbf{L}(X_t, \pi_t) + \Omega(\widehat{\boldsymbol{\mathcal{L}}}'_t \mathbf{C}(X_t, \pi_t)) \frac{\widehat{\epsilon}_t^m}{\Omega(\widehat{\boldsymbol{\mathcal{L}}}'_t \mathbf{C}(X_t^m, \pi_t^m))}\right).$$

Here, it is also common to replace $\widehat{\boldsymbol{\Lambda}}_t$ with the weighted least squares estimator that can be found after estimation of $\Omega(\cdot)$.

LSMC Algorithm

We utilise a discretised version of Kharroubi et al. (2015) with some modifications in forward asimulation.

Algorithm 1 Forward simulation

1: **for** $t = 0$ **to** $N - 1$ **do**
2:     **for** $m = 1$ **to** M **do**
        [Simulate random samples]
3:         $X_t^m := Rand \in \mathcal{X}$ ▷ State
4:         $\widetilde{\pi}_t^m := Rand \in \mathcal{A}$ ▷ Control
5:         $z_{t+1}^m := Rand \in \mathcal{Z}$ ▷ Disturbance
        [Compute the state variable after control]
6:         $\widetilde{X}_{t+1}^m := \mathcal{T}_t(X_t^m, \widetilde{\pi}_t^m, z_{t+1}^m)$ ▷ Evolution of state
7:     **end for**
8: **end for**

Algorithm 2 Backward solution (Realised value)

for $t = N$ **to** 0 **do**

  if $t = N$ **then** $\widehat{V}_t(\widetilde{\boldsymbol{X}}_t) := R_N(\widetilde{\boldsymbol{X}}_t)$

  else if $t < N$ **then**

    [Regression of transformed value function]

    $\widehat{\boldsymbol{\Lambda}}_t := \arg\min_{\boldsymbol{\Lambda}_t} \sum_{m=1}^{M} \left[\boldsymbol{\Lambda}_t'\mathbf{L}(X_t^m, \widetilde{\pi}_t) - H^{-1}(\beta \widehat{V}_{t+1}(\widetilde{X}_{t+1}^m))\right]^2$

    [Approximate conditional expectation] $\widehat{\Phi}_t(X_t, \widetilde{\pi}_t) := H^B(\widehat{\boldsymbol{\Lambda}}_t'\mathbf{L}(X_t, \widetilde{\pi}_t))$

    for $m = 1$ **to** M **do**

      $\widehat{X}_t^m := \widetilde{X}_t^m$

      [Optimal control] $\pi_t^*(\widehat{X}_t^m) := \arg\sup_{\pi_t \in A} \left\{R_t(\widehat{X}_t^m, \pi_t) + \widehat{\Phi}_t(\widehat{X}_t^m, \pi_t)\right\}$

      [Update value function with optimal paths]

      $\widehat{V}_t(\widehat{X}_t^m) := R_t(\widehat{X}_t^m, \pi_t^*(\widehat{X}_t^m))$

      $\widehat{X}_{t+1}^m := \mathcal{T}_t(\widehat{X}_t^m, \pi_t^*(\widehat{X}_t^m), z_t^m)$

      for $t_j = t + 1$ **to** $N - 1$ **do**

        $\widehat{V}_t(\widehat{X}_t^m) := \widehat{V}_t(\widehat{X}_t^m) + \beta^{t_j - t} R_{t_j}(\widehat{X}_{t_j}^m, \pi_{t_j}^*(\widehat{X}_{t_j}^m))$

        $\widehat{X}_{t_j+1}^m := \mathcal{T}_t(\widehat{X}_{t_j}^m, \pi_{t_j}^*(\widehat{X}_{t_j}^m), z_{t_j}^m)$

      end for

      $\widehat{V}_t(\widehat{X}_t^m) := \widehat{V}_t(\widehat{X}_t^m) + \beta^{N-t} R_N(\widehat{X}_N^m)$

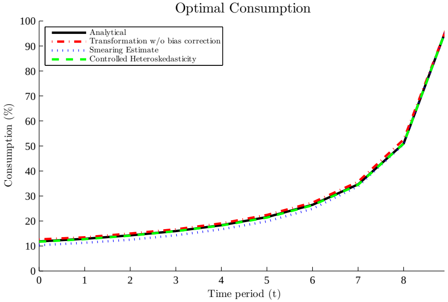
    end for

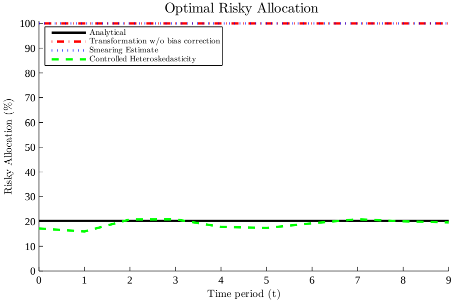
  end if

end for

Simple multi-period CRRA utility model where $R_t(x) = G_T(x) = \frac{1}{\gamma}x^{\gamma}$. Agent optimises consumption and risky asset allocation each period, $\pi_t = (\alpha_t, \delta_t) \in [0,1] \times [0,1]$. $\gamma = -10$, endogenous state variable wealth X_t grows between periods based on a stochastic return $Z_t \sim \mathcal{N}(0.1, 0.2)$ and deterministic rate $r = 0.03$ with transition function

$$T(X_t^{\pi}, \pi_t, z_{t+1}) = X_t^{\pi}(1-\alpha_t)e^{\delta_t z_{t+1} + (1-\delta_t)r}. \tag{23}$$





Retirement Model Extensions

Introduce stochastic real interest rate as a Vasicek process. Yearly discretised and simulated with

$$r_{t+1} = \bar{r} + e^{-b}(r_t - \bar{r}) + \sqrt{\frac{\sigma_R^2}{2b}(1 - e^{-2b})}\epsilon_{t+1}, \quad \epsilon_t \overset{i.i.d}{\sim} \mathcal{N}(0,1), \quad (24)$$

where $\bar{r} \in \mathbb{R}^+$ is the long term mean, $b \in (0,1]$ the speed of adjustments and σ_R the volatility.

Introduce a separate taxable deposit account $\widetilde{W}_t \in \mathbb{R}^+$, such that $C_t = \alpha_t(W_t + \widetilde{W}_t) + P_t$. Always preferred for spending over liquid wealth. Same dynamics and assumptions as liquid wealth.

Model Extension: deposit account

Let ν_t be the minimum withdrawal rate.
If deposit account $\widetilde{W}_t$ is large enough to cover any consumption above the minimum withdrawal from pension account i.e.
$\widetilde{W}_t(1-\alpha_t) > W_t(\alpha_t - \nu_t)$, then

$$\begin{aligned} W_{t+1} =& W_t(1-\nu_t) \times (\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}), \\ \widetilde{W}_{t+1} =& [\widetilde{W}_t(1-\alpha_t) + W_t(\nu_t - \alpha_t)] \times (\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}) \\ & - \Theta(\widetilde{W}_t(1-\alpha_t) + W_t(\nu_t-\alpha_t), \delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}), \end{aligned} \tag{25}$$

where $\Theta(w,z) = 0.15w\max(z-1.0)$ is the tax function.
if $\widetilde{W}_t(1-\alpha_t) \leq W_t(\alpha_t - \nu_t)$, then

$$\begin{aligned} W_{t+1} =& (W_t + \widetilde{W}_t)(1-\alpha_t) \times (\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}), \\ \widetilde{W}_{t+1} =& 0. \end{aligned} \tag{26}$$

Model Extension 1 - Annuities

- The retiree can at any time chose part $\iota_t \in [0, 1-\alpha_t]$ of total liquid wealth $(\widetilde{W}_t + W_t)$ to annuitise, where new state variable $Y_t \in \mathcal{Y} = \mathbb{R}^+$ holds annuity payments.
- Annuity-immediate, hence payments start at $t+1$ and priced to have constant payments in real terms.
- Any annuitisation is reflected in consumption $C_t = \alpha_t(W_t + \widetilde{W}_t) + P_t + Y_t$.
- **Fairly priced with risk-neutral term structure from calibrated Vasicek model**, with actuarial present value

$$\ddot{a}(y,t) = \sum_{i=t+1}^{T} {}_t p_i J(t,i,y) \tag{27}$$

where y is annuity payment, $J(t,i,j)$ a zero coupon bond with maturity i and face value y, and ${}_t p_i$ the probability to survive from year t to i.

Model Extension 1 - Annuities

- Denote a state vector as $X_t = (W_t, \widetilde{W}_t, G_t, H_t, r_t, Y_t) \in \mathcal{W} \times \mathcal{W} \times \mathcal{G} \times \mathcal{H} \times \mathcal{R} \times \mathcal{Y}$, where $\mathcal{W}, \mathcal{H}, \mathcal{R}, \mathcal{Y} = \mathbb{R}^+$ and $\mathcal{G} = \{\Delta, 0, 1, 2\}$.
- Denote an action space of $(\alpha_t, \delta_t, \iota_t, \varrho) \in \mathcal{A} = (-\infty, 1] \times [0,1] \times [0,1] \times \{0, [H_L, \mathrm{W}]\}$ for $t = t_0$, and $(\alpha_t, \delta_t, \iota_t) \in \mathcal{A} = (-\infty, 1] \times [0,1] \times [0,1]$ for $t = t_0 + 1, ..., T-1$.
- Denote an admissible space of state-action combination as $D_t(x_t) = \left\{ \pi_t(x_t) \in \mathcal{A} \;\middle|\; \alpha_t \geq \frac{\overline{c}_d - P_t - Y_t}{W_t + \widetilde{W}_t},\ \alpha_t + \iota_t \leq 1 \right\}$ which contains the possible actions for the current state.
- Transition functions exist for state variables W_t, $\widetilde{W}_t$, Y_t and r_t. Define the total transition function

$$T_t(W_t, \widetilde{W}_t, Y_t, r_t, \alpha_t, \delta_t, y, z_{t+1}) = \begin{bmatrix} T_t^W(W_t, \alpha_t, \delta_t, z_{t+1}, r_{t+1}) \\ T_t^{\widetilde{W}}(\widetilde{W}_t, \alpha_t, \delta_t, z_{t+1}, r_{t+1}) \\ T_t^Y(Y_t, y) \\ T_t^r(r_t) \end{bmatrix}$$

Model Extension 1 - Annuities

$$T_t^W(\cdot) := \begin{cases} W_t(1-\alpha_t-\iota_t) & \text{if } \widetilde{W}_t(1-\alpha_t-\iota_t) > \\ \quad \times\left(\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}\right), & W_t(\alpha_t+\iota_t-\nu), \\ (W_t+\widetilde{W}_t)(1-\alpha_t-\iota_t) & \\ \quad \times\left(\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}\right), & \text{otherwise.} \end{cases} \tag{28}$$

$$T_t^{\widetilde{W}}(\cdot) := \begin{cases} \left(\widetilde{W}_t(1-\alpha_t-\iota_t) + W_t(\nu-\alpha_t-\iota_t)\right) & \\ \quad \times\left(\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}\right) & \text{if } \widetilde{W}_t(1-\alpha_t-\iota_t) > \\ \quad -\Theta\left(\widetilde{W}_t(1-\alpha_t-\iota_t) + W_t(\nu-\alpha_t-\iota_t),\right. & W_t(\alpha_t+\iota_t-\nu), \\ \left.\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}\right), & \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

$$T_t^Y(Y_t, y) := Y_{t+1} = Y_t + y, \tag{30}$$

$$T_t^r(r_t) := r_{t+1} = r_t + b(\bar{r} - r_t) + \sigma_R \epsilon_{t+1} \tag{31}$$

Value of annuity included in means-test, with a linear value decrease each year. Approximated by pricing an annuity with same annuity payments.

Function for income test

$$P_{\mathrm{I}} := P^{d}_{\max} - \left(P_{\mathrm{D}}(W_t) + y - \frac{\ddot{a}(y,t)}{T-t} - L^{d}_{\mathrm{I}} \right) \varpi^{d}_{\mathrm{I}}. \tag{32}$$

Function for the asset-test

$$P_{\mathrm{A}} := P^{d}_{\max} - \left(W_t + \ddot{a}(y,t) - L^{d,h}_{\mathrm{A}} \right) \varpi^{d}_{\mathrm{A}}. \tag{33}$$

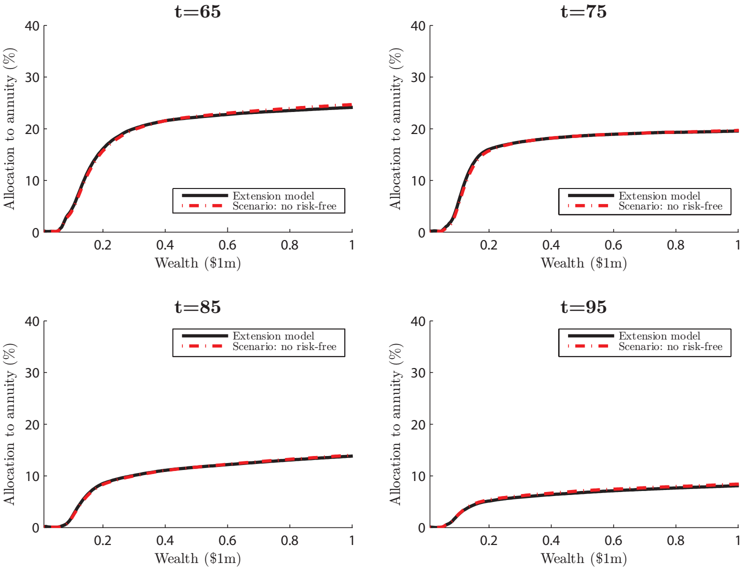


Figure: Optimal annuitisation given initial liquid wealth and age (no prior annuitisation).

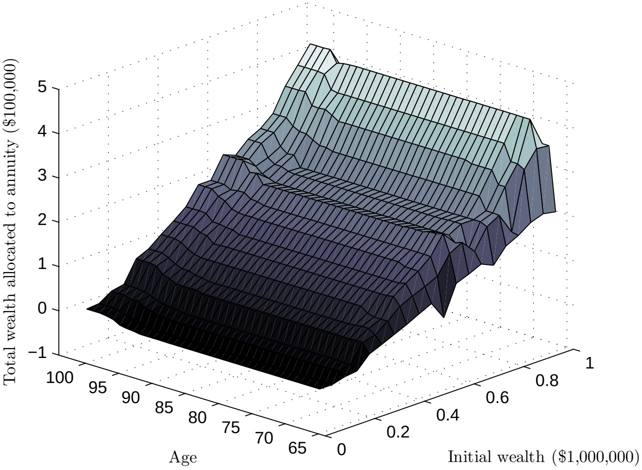


Figure: Optimal cumulative annuitisation over time given initial liquid wealth.

Conclusions - Extension 1

- It is optimal to annuitise earlier rather than later in retirement (due to the mortality credit). The exception is for very poor households.
- Delaying annuitisation leads to less wealth annuitised, but higher annuity payments.
- The means-test decreases the ‘demand’ for annuities, but does not eliminate it. Retiree with low likelihood to access Age Pension has constant annuitisation rate.
- The mortality credit from the annuity dominates the utility received from bequeathing this wealth. **Optimal annuitisation is the same with/out access to a risk-free rate**.

Model Extension 2 - Flexible housing

Australian retirees are 'house rich, but asset poor', and can optimise Age Pension payments by overallocating to the family home. We extend the model to flexible housing decisions by scaling housing and access to a reverse mortgage.

Reverse mortgage:

- Loan against the home equity up to an age dependent loan-to-value ratio, with no amortisation/interest payments required.
- Starts at 20-25% at age 65, and increases 1% per year.
- Multiple options how to access: lump sum, credit line, tenure etc.
- Interest and fees accumulate, but is capped by the house value.
- At death (or sale of home) the loan is paid off, and any equity remaining is returned.

Model Extension 2 - Flexible housing

- The retiree can at any time up or downscale housing with a proportion $\tau_t \in [-1, \infty]$, to get a new house valued $H_t(1+\tau_t)$.
- **If $\tau_t \neq 0$, a proportional transaction cost $\o$ applies to the current house value.** Taxes are not modelled.
- The retiree can at any time chose a proportion $l_t \in [0, \frac{\overline{L}_t}{H_t}]$ up to loan-to-value threshold $\overline{L}_t$ as a reverse mortgage from the home value, which adds to the outstanding loan state $L_t \in \mathcal{L} = \mathbb{R}^+$.
- The loan-to-value ratio is time dependent and defined as a proportion of the home value $\overline{L}_t = H_t l(t)$ where

$$l(t) = 0.2 + 0.01(\min(85, t) - 65). \quad (34)$$

Model Extension 2 - Flexible housing

- Denote a state vector as $X_t = (W_t, \widetilde{W}_t, G_t, H_t, r_t, L_t) \in \mathcal{W} \times \mathcal{W} \times \mathcal{G} \times \mathcal{H} \times \mathcal{R} \times \mathcal{L}$, where $\mathcal{W}, \mathcal{H}, \mathcal{R}, \mathcal{L} = \mathbb{R}^+$ and $\mathcal{G} = \{\Delta, 0, 1, 2\}$.
- Denote an action space of $(\alpha_t, \delta_t, \varrho) \in \mathcal{A} = (-\infty, 1] \times [0, 1] \times \{0, [H_L, \mathrm{W}]\}$ for $t = t_0$, and $(\alpha_t, \delta_t, \tau_t, l_t) \in \mathcal{A} = (-\infty, 1] \times [0, 1] \times [-1, \infty] \times [0, \frac{\overline{L}_t}{H_t}]$ for $t = t_0 + 1, ..., T - 1$.
- The admissible state space $D_t(x_t)$ requires additional constraints.

  Loan restrictions: $l_t \leq \max\left(0, \overline{L}_t - \frac{L_t \mathbb{I}_{\tau=0}}{H_t(1+\tau_t)}\right)$,

  scaling restrictions $\tau_t \leq \frac{W_t + \widetilde{W}_t - \mathbb{I}_{\tau\neq 0}\{\o H_t + L_t\}}{H_t}$,

  and budget constraint

$$\alpha_t(W_t + \widetilde{W}_t) + \mathbb{I}_{\tau\neq 0}\{\o H_t + L_t\} - l_t H_t(1 + \tau_t) - (W_t + \widetilde{W}_t) \leq 0.$$

- Transition functions are needed for state variables W_t, $\widetilde{W}_t$, H_t, L_t and r_t. Define the total transition function

$$T_t(W_t, \widetilde{W}_t, L_t, H_t, r_t, \alpha_t, \delta_t, \tau_t, l_t, z_{t+1}, r_{t+1}) = \begin{bmatrix} T_t^W(W_t, \alpha_t, \delta_t, z_{t+1}, r_{t+1}) \\ T_t^{\widetilde{W}}(\widetilde{W}_t, \alpha_t, \delta_t, z_{t+1}, r_{t+1}) \\ T_t^H(H_t, \tau_t) \\ T_t^L(L_t, H_t, l_t, \tau_t, r_{t+1}) \\ T_t^r(r_t) \end{bmatrix}$$

Model Extension 2 - Flexible housing

Let $\Delta_t^H = \frac{l_t H_t(1+\tau_t) - \mathbb{I}_{\tau_t \neq 0}\{H_t(\tau_t + \o) + L_t\}}{W_t + \widetilde{W}_t}$, then the transition becomes

$$T_t^W(\cdot) := \begin{cases} W_t(1-\alpha_t-\Delta_t^H) \\ \quad \times (\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}), & \text{if } \widetilde{W}_t(1-\alpha_t-\Delta_t^H) >, \\ & W_t(\alpha_t+\Delta_t^H-\nu) \\ (W_t+\widetilde{W}_t)(1-\alpha_t-\Delta_t^H) & \text{otherwise,} \\ \quad \times (\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}), \end{cases} \tag{35}$$

$$T_t^{\widetilde{W}}(\cdot) := \begin{cases} (\widetilde{W}_t(1-\alpha_t-\Delta_t^H) + W_t(\nu_t-\alpha_t-\Delta_t^H)) \\ \quad \times (\delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}) & \text{if } \widetilde{W}_t(1-\alpha_t-\Delta_t^H) >, \\ \quad - \Theta(\widetilde{W}_t(1-\alpha_t-\Delta_t^H) + W_t(\nu-\alpha_t-\Delta_t^H), & W_t(\alpha_t+\Delta_t^H-\nu_t) \\ \delta_t e^{z_{t+1}} + (1-\delta_t)e^{r_{t+1}}), \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

$$T_t^L(\cdot) := L_{t+1} = (L_t \mathbb{I}_{\tau_t=0} + l_t H_t(1+\tau_t))e^{r_{t+1}+\varphi}, \tag{37}$$

$$T_t^H(\cdot) := H_{t+1} = H_t(1+\tau_t). \tag{38}$$

we set cost of selling house: $\o = 6\%$, interest rate markup: $\varphi = 0.0242$; and risk-free rate parameters: $b = 0.64$, $\bar{r} = 0.013$ and $\sigma_R = 0.016$.

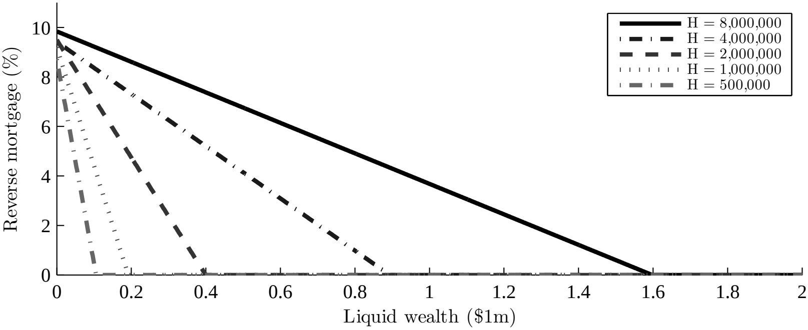


Figure: Optimal proportion reverse mortgage.

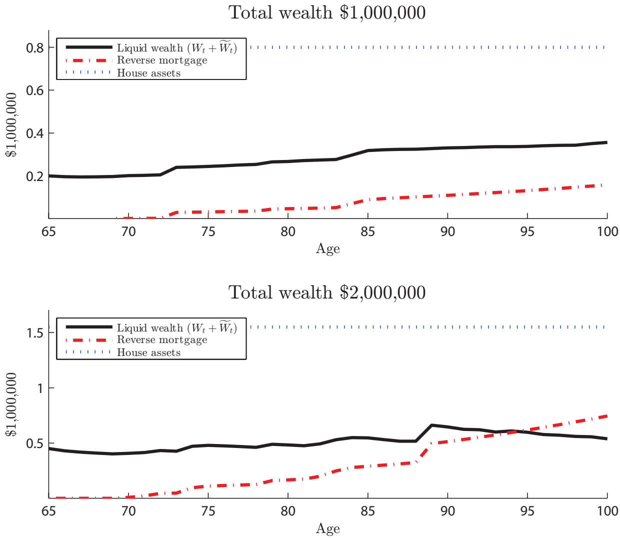


Figure: Wealth, house and reverse mortgage paths in retirement given low, medium and high initial total wealth.

Results - Extension 2

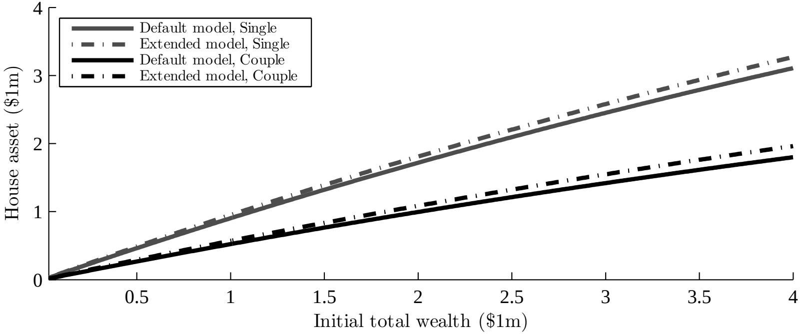


Figure: Optimal allocation to housing at retirement for the default case compared to extension model 2 where decisions for scaling housing and reverse mortgage are available.

Conclusions - Extension 2

- The proportion reverse mortgage increases with house value and decreases with wealth (confirms empirical results in Chiang and Tsai (2016)).
- The proportion increases with age up to 80, and then flattens. Never reaches the LVR.
- **It is never optimal to downsize housing, even when overallocated, unless certain events are incurring significant costs.**
- Scaling housing is more costly than reverse mortgages for accessing part of home equity. In addition, a reverse mortgage allows the retiree to still receive utility from the larger home.
- Only marginal effect on initial housing allocation with the additional control variables.

Thank you for your attention!

* I'm not a financial advisor - so don't hold me accountable for advice!

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